Cleo Bench Problem 15

Evaluate $\int_0^1 \frac{\ln(1-x)}{x}\mathrm{Li}_3\left(\frac{1+x}{2}\right)dx$, $\int_0^1 \frac{\ln^2(1-x)}{x}\mathrm{Li}_2\left(\frac{1+x}{2}\right)dx$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I_1 = \int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{Li}_3\!\left(\frac{1+x}{2}\right)dx, \qquad I_2 = \int_0^1 \frac{\ln^2(1-x)}{x}\,\mathrm{Li}_2\!\left(\frac{1+x}{2}\right)dx.$$

Result

With $L = \ln 2$, $\zeta_s = \zeta(s)$, and $\mathrm{Li}_s(\frac{1}{2})$ the polylogarithm at $\frac{1}{2}$:

$$\boxed{I_1 = \frac{29}{16}\zeta_5 - \frac{19}{16}\zeta_2\zeta_3 + \frac{11}{16}\zeta_4 L + \frac{5}{16}\zeta_3 L^2 - \frac{5}{12}\zeta_2 L^3 + \frac{L^5}{40} - 3\,\mathrm{Li}_5\!\left(\tfrac{1}{2}\right)}$$

$$\boxed{I_2 = \frac{81}{32}\zeta_5 + \frac{15}{8}\zeta_2\zeta_3 - 6\zeta_4 L - \frac{1}{8}\zeta_3 L^2 - \frac{1}{3}\zeta_2 L^3 + \frac{L^5}{15} + 2L\,\mathrm{Li}_4\!\left(\tfrac{1}{2}\right) + 2\,\mathrm{Li}_5\!\left(\tfrac{1}{2}\right)}$$

Equivalently, in π-notation ($\zeta_2 = \pi^2/6,\ \zeta_4 = \pi^4/90$):

$$\begin{aligned} I_1 &= \frac{29}{16}\zeta(5) - \frac{19\pi^2}{96}\zeta(3) + \frac{11\pi^4}{1440}\ln 2 \\ &\quad + \frac{5}{16}\zeta(3)\ln^2 2 - \frac{5\pi^2}{72}\ln^3 2 + \frac{\ln^5 2}{40} - 3\,\mathrm{Li}_5\!\left(\tfrac{1}{2}\right), \\ I_2 &= \frac{81}{32}\zeta(5) + \frac{5\pi^2}{16}\zeta(3) - \frac{\pi^4}{15}\ln 2 \\ &\quad - \frac{1}{8}\zeta(3)\ln^2 2 - \frac{\pi^2}{18}\ln^3 2 + \frac{\ln^5 2}{15} \\ &\quad + 2\ln 2\,\mathrm{Li}_4\!\left(\tfrac{1}{2}\right) + 2\,\mathrm{Li}_5\!\left(\tfrac{1}{2}\right). \end{aligned}$$

Numerically

$$\begin{aligned} I_1 &= -1.5218229288558100152583924848553931153302856804186 6\ldots \\ I_2 &= 3.3209668786648676803459861212048945096771586843076 6\ldots \end{aligned}$$

Notice that $\mathrm{Li}_4(\frac{1}{2})$ drops out of I_1; both answers lie in the standard weight-5 "level 2" space spanned by $\{\zeta_5,\ \zeta_2\zeta_3,\ \zeta_4 L,\ \zeta_3 L^2,\ \zeta_2 L^3,\ L^5,\ L\,\mathrm{Li}_4(\frac{1}{2}),\ \mathrm{Li}_5(\frac{1}{2})\}$.

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[*] Problem originally posed on Mathematics Stack Exchange (question 970125, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Derivation

Throughout, $u, t, a \in (0,1)$, $L = \ln 2$, $H_m^{(p)} = \sum_{k\le m} k^{-p}$, $H_m = H_m^{(1)}$. All series manipulations below involve either (i) power series with radius $>$ the domain used, or (ii) non-negative terms (after writing $\zeta_p - H_m^{(p)} = \sum_{j>m} j^{-p}$), so all interchanges of summation/integration are justified by absolute convergence / monotone convergence; the few Abel-limit arguments are flagged where they occur. Every displayed identity was in addition verified numerically to ≥ 40 significant digits.

0. Plan and toolbox

Substituting $u = 1 - x$,

$$I_1 = \int_0^1 \frac{\ln u}{1-u}\,\mathrm{Li}_3\Big(1 - \frac{u}{2}\Big)\,du, \qquad I_2 = \int_0^1 \frac{\ln^2 u}{1-u}\,\mathrm{Li}_2\Big(1 - \frac{u}{2}\Big)\,du. \tag{0.1}$$

Both integrals will be reduced, by elementary and fully justified steps, to **multiple polylogarithms at $\frac{1}{2}$**,

$$\mathrm{Li}_{s_1,\ldots,s_k}\Big(\tfrac{1}{2}\Big) := \sum_{n_1>n_2>\cdots>n_k\ge 1} \frac{2^{-n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

together with ordinary polylogarithms at $\frac{1}{2}$. Section 3 evaluates *all* such constants of weight ≤ 5 in closed form by a provable linear system of relations (this is the hard core: the weight-5 constants X, Y, W, V defined below). Sections 1–2 perform the reductions; Section 4 assembles the answers; Section 5 records independent cross-checks.

Standard moment formulas, obtained by expanding $\frac{1}{1-u} = \sum_{r\ge 0} u^r$ and using $\int_0^1 u^q \ln^j u\,du = (-1)^j j!/(q+1)^{j+1}$:

$$\int_0^1 \frac{u^m \ln u}{1-u}du = -(\zeta_2 - H_m^{(2)}), \quad \int_0^1 \frac{u^m \ln^2 u}{1-u}du = 2(\zeta_3 - H_m^{(3)}),$$
$$\int_0^1 \frac{u^m \ln^3 u}{1-u}du = -6(\zeta_4 - H_m^{(4)}). \tag{0.2}$$

Elementary antiderivatives used repeatedly (differentiate to check; constants fixed at the endpoints):

$$\int_0^x \frac{\ln^2(1-t)}{t}dt = \ln^2(1-x)\ln x + 2\ln(1-x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta_3, \tag{0.3}$$

$$\int_0^x \frac{\ln(1-t)\,\mathrm{Li}_2(t)}{t}\,dt = -\tfrac{1}{2}\,\mathrm{Li}_2(x)^2 \qquad \Big(\text{since } d\,\mathrm{Li}_2(t) = -\tfrac{\ln(1-t)}{t}dt\Big), \tag{0.4}$$

$$\frac{d}{dt}\,\mathrm{Li}_2(1-t) = \frac{\ln t}{1-t}, \qquad \frac{d}{dt}\,\mathrm{Li}_3(1-t) = -\frac{\mathrm{Li}_2(1-t)}{1-t}. \tag{0.5}$$

Euler's reflection (proved by differentiating both sides and checking at $x = 1$):

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta_2 - \ln x \ln(1-x). \tag{0.6}$$

Special values (the first is (0.6) at $x = \frac{1}{2}$; the second is Landen's classical trilogarithm evaluation, re-derived independently by the engine of Section 3, see (3.9)):

$$\mathrm{Li}_2(\tfrac{1}{2}) = \tfrac{\zeta_2}{2} - \tfrac{L^2}{2}, \qquad \mathrm{Li}_3(\tfrac{1}{2}) = \tfrac{7}{8}\zeta_3 - \tfrac{\zeta_2 L}{2} + \tfrac{L^3}{6}. \tag{0.7}$$

Four weight-5 constants play the central role:

$$X=\sum_{n\ge1}\frac{H_n}{2^n n^4},\quad Y=\sum_{n\ge1}\frac{H_n^{(2)}}{2^n n^3},\quad W=\int_0^{1/2}\frac{\mathrm{Li}_2(t)^2}{t}dt,\quad V=\int_0^{1/2}\frac{\ln^2(1-t)\,\mathrm{Li}_2(t)}{t}dt. \tag{0.8}$$

Expanding $\mathrm{Li}_2(t)^2=\sum_n t^n\big(\frac{2H_{n-1}^{(2)}}{n^2}+\frac{4H_{n-1}}{n^3}\big)$ (Cauchy product plus the partial fraction $\frac{1}{a^2b^2}=\frac{1}{(a+b)^2}\big(\frac{1}{a^2}+\frac{1}{b^2}\big)+\frac{2}{(a+b)^3}\big(\frac{1}{a}+\frac{1}{b}\big)$ summed over $a+b=n$) and integrating termwise,

$$W=2Y+4X-6\,\mathrm{Li}_5(\tfrac12). \tag{0.9}$$

1. Reduction of I_2

Apply (0.6) with $x=u/2$ inside (0.1):

$$\mathrm{Li}_2\Big(1-\frac{u}{2}\Big)=\zeta_2-\big(\ln u-L\big)\ln\Big(1-\frac{u}{2}\Big)-\mathrm{Li}_2\Big(\frac{u}{2}\Big).$$

Using $\int_0^1\frac{\ln^2 u}{1-u}du=2\zeta_3$, the series $\ln(1-\frac{u}{2})=-\sum_m\frac{u^m}{2^m m}$, $\mathrm{Li}_2(\frac{u}{2})=\sum_m\frac{u^m}{2^m m^2}$ and the moments (0.2):

$$I_2=2\zeta_2\zeta_3-6\sum_{m\ge1}\frac{\zeta_4-H_m^{(4)}}{2^m m}-2L\sum_{m\ge1}\frac{\zeta_3-H_m^{(3)}}{2^m m}-2\sum_{m\ge1}\frac{\zeta_3-H_m^{(3)}}{2^m m^2}. \tag{1.1}$$

The ζ-parts sum to $\zeta_4 L,\ \zeta_3 L,\ \zeta_3\,\mathrm{Li}_2(\frac12)$. For the harmonic parts define

$$\sigma_b=\sum_m\frac{H_m^{(3)}}{2^m m},\qquad \sigma_c=\sum_m\frac{H_m^{(3)}}{2^m m^2},\qquad \sigma_a=\sum_m\frac{H_m^{(4)}}{2^m m}.$$

Lemma 1.1. For $0<x<1$:

$$\sum_m\frac{H_m^{(3)}}{m}x^m=\mathrm{Li}_4(x)-\ln(1-x)\mathrm{Li}_3(x)-\tfrac12\mathrm{Li}_2(x)^2.$$

Proof. $\sum_m H_m^{(3)}x^m=\frac{\mathrm{Li}_3(x)}{1-x}$; divide by x and integrate: $\int_0^x\frac{\mathrm{Li}_3(t)}{t(1-t)}dt=\mathrm{Li}_4(x)+\int_0^x\frac{\mathrm{Li}_3(t)}{1-t}dt$, and by parts with (0.4), $\int_0^x\frac{\mathrm{Li}_3}{1-t}dt=-\ln(1-x)\mathrm{Li}_3(x)+\int_0^x\frac{\ln(1-t)\mathrm{Li}_2(t)}{t}dt=-\ln(1-x)\mathrm{Li}_3(x)-\frac12\mathrm{Li}_2(x)^2$. □

At $x=\frac12$: $\sigma_b=\mathrm{Li}_4(\frac12)+L\,\mathrm{Li}_3(\frac12)-\frac12\mathrm{Li}_2(\frac12)^2$.

Lemma 1.2. $\int_0^x\frac{\ln(1-t)\,\mathrm{Li}_3(t)}{t}dt=-\mathrm{Li}_2(x)\mathrm{Li}_3(x)+\int_0^x\frac{\mathrm{Li}_2(t)^2}{t}dt$ (integrate by parts using (0.4)); hence, dividing Lemma 1.1 by x and integrating to $\frac12$,

$$\sigma_c=\mathrm{Li}_5(\tfrac12)+\mathrm{Li}_2(\tfrac12)\mathrm{Li}_3(\tfrac12)-\tfrac32\,W.$$

Lemma 1.3. Analogously $\sum_m\frac{H_m^{(4)}}{m}x^m=\mathrm{Li}_5(x)-\ln(1-x)\mathrm{Li}_4(x)+\int_0^x\frac{\ln(1-t)\mathrm{Li}_3(t)}{t}dt$, so by Lemma 1.2,

$$\sigma_a=\mathrm{Li}_5(\tfrac12)+L\,\mathrm{Li}_4(\tfrac12)-\mathrm{Li}_2(\tfrac12)\mathrm{Li}_3(\tfrac12)+W.$$

Substituting the three lemmas into (1.1) and using (0.7):

$$\boxed{\begin{aligned}I_2=2\zeta_2\zeta_3-6\zeta_4L-2\zeta_3L^2-2\zeta_3\,\mathrm{Li}_2(\tfrac12)+8\,\mathrm{Li}_5(\tfrac12)+8L\,\mathrm{Li}_4(\tfrac12)\\ -4\,\mathrm{Li}_2(\tfrac12)\mathrm{Li}_3(\tfrac12)+2L^2\mathrm{Li}_3(\tfrac12)-L\,\mathrm{Li}_2(\tfrac12)^2+3W\end{aligned}} \tag{1.2}$$

— i.e. I_2 is *explicit polylogarithms at* $\frac12$ *plus* $3W$. (Identity (1.2) checked numerically to 70 digits.)

2. Reduction of I_1

2.1 The parameter representation. By the fundamental theorem of calculus in the parameter a,

$$\mathrm{Li}_3\Big(1-\frac{u}{2}\Big)=\zeta_3-\int_0^{1/2}\frac{u\,\mathrm{Li}_2(1-au)}{1-au}\,da,$$

so by (0.1), Fubini (the integrand is $\ln u\cdot$ (bounded positive), absolutely integrable), and the partial fraction $\frac{u}{(1-u)(1-au)}=\frac{1}{1-a}\big[\frac{1}{1-u}-\frac{1}{1-au}\big]$:

$$\begin{gathered}I_1=-\zeta_2\zeta_3-\int_0^{1/2}\frac{da}{1-a}\Big[\mathfrak{A}(a)-\mathfrak{B}(a)\Big],\\ \mathfrak{A}(a)=\int_0^1\frac{\ln u\,\mathrm{Li}_2(1-au)}{1-u}du,\quad \mathfrak{B}(a)=\int_0^1\frac{\ln u\,\mathrm{Li}_2(1-au)}{1-au}du.\end{gathered}\tag{2.1}$$

2.2 $\mathfrak{B}$ in closed form. Substituting $s=au$ and using the antiderivatives $\int\frac{\mathrm{Li}_2(1-s)}{1-s}ds=-\mathrm{Li}_3(1-s)$ and, by (0.5) and (0.4) with $r=1-s$, $\int\frac{\ln s\,\mathrm{Li}_2(1-s)}{1-s}ds=\frac{1}{2}\mathrm{Li}_2(1-s)^2$:

$$\mathfrak{B}(a)=\frac{1}{a}\Big[\tfrac{1}{2}\mathrm{Li}_2(1-a)^2-\tfrac{1}{2}\zeta_2^2-\ln a\,(\zeta_3-\mathrm{Li}_3(1-a))\Big].\tag{2.2}$$

2.3 $\mathfrak{A}$ as a series. Apply (0.6) to $\mathrm{Li}_2(1-au)$, expand $\ln(1-au)$, $\mathrm{Li}_2(au)$, and use (0.2):

$$\mathfrak{A}(a)=-\zeta_2^2-\ln a\sum_{m\geq1}\frac{a^m}{m}(\zeta_2-H_m^{(2)})+2\sum_{m\geq1}\frac{a^m}{m}(\zeta_3-H_m^{(3)})+\sum_{m\geq1}\frac{a^m}{m^2}(\zeta_2-H_m^{(2)}).\tag{2.3}$$

2.4 Integrating $\mathfrak{A}$. With the tails $\lambda_m=\sum_{i>m}\frac{2^{-i}}{i}$, $\mu_m=\sum_{i>m}\frac{2^{-i}}{i^2}$ one has the elementary moments

$$\int_0^{1/2}\frac{a^m}{1-a}da=\lambda_m,\qquad\int_0^{1/2}\frac{a^m\ln a}{1-a}da=-(L\lambda_m+\mu_m),$$

(expand $\frac{1}{1-a}$ geometrically; $\int_0^{1/2}a^{i-1}\ln a\,da=-2^{-i}(L/i+1/i^2)$). Hence

$$-\int_0^{1/2}\frac{\mathfrak{A}(a)}{1-a}da=\zeta_2^2L-L\,\mathfrak{T}_1-\mathfrak{T}_2-2\,\mathfrak{T}_3-\mathfrak{T}_4,\tag{2.4}$$

$$\begin{aligned}\mathfrak{T}_1&=\sum_m\frac{(\zeta_2-H_m^{(2)})\lambda_m}{m},&\mathfrak{T}_2&=\sum_m\frac{(\zeta_2-H_m^{(2)})\mu_m}{m},\\ \mathfrak{T}_3&=\sum_m\frac{(\zeta_3-H_m^{(3)})\lambda_m}{m},&\mathfrak{T}_4&=\sum_m\frac{(\zeta_2-H_m^{(2)})\lambda_m}{m^2}.\end{aligned}$$

2.5 The $\mathfrak{T}$'s are multiple polylogarithms at $\frac{1}{2}$. Write $\zeta_p-H_m^{(p)}=\sum_{j>m}j^{-p}$ and $\lambda_m=\sum_{i>m}2^{-i}/i$; the resulting triple sums have all terms ≥0, so they may be rearranged freely. Splitting the region $\{i>m,\ j>m\}$ into $\{i>j>m\}\cup\{j=i>m\}\cup\{j>i>m\}$ and, in the last region, resumming $\sum_{j>i}j^{-p}=\zeta_p-H_i^{(p)}$ and iterating the same split once more, every piece becomes a *descending chain* with the 2-power on the outermost index — that is, a multiple polylogarithm at $\frac{1}{2}$. The result (each line verified to 45 digits):

$$\begin{aligned}\mathfrak{T}_1&=\zeta_2\,\mathrm{Li}_{1,1}-\mathrm{Li}_{1,1,2}-\mathrm{Li}_{1,3},&\mathfrak{T}_2&=\zeta_2\,\mathrm{Li}_{2,1}-\mathrm{Li}_{2,1,2}-\mathrm{Li}_{2,3},\\ \mathfrak{T}_3&=\zeta_3\,\mathrm{Li}_{1,1}-\mathrm{Li}_{1,1,3}-\mathrm{Li}_{1,4},&\mathfrak{T}_4&=\zeta_2\,\mathrm{Li}_{1,2}-\mathrm{Li}_{1,2,2}-\mathrm{Li}_{1,4},\end{aligned}\tag{2.5}$$

all Li's at argument $\frac{1}{2}$. (E.g. $\mathfrak{T}_1=\sum_i\frac{2^{-i}}{i}\sum_{m<i}\frac{\zeta_2-H_m^{(2)}}{m}$ and $\sum_{m<i}\frac{\zeta_2-H_m^{(2)}}{m}=\zeta_2H_{i-1}-\sum_{k<m<i}\frac{1}{mk^2}-\sum_{m<i}\frac{1}{m^3}$.)

2.6 The $\mathfrak{B}$-part. With $1/(a(1-a)) = 1/a + 1/(1-a)$,

$$\Xi := \int_0^{1/2} \frac{\mathfrak{B}(a)}{1-a} da = \tfrac{1}{2}W_3' + \tfrac{1}{2}(S - W) - \tfrac{1}{2}\zeta_2^2 L - \alpha - \beta, \tag{2.6}$$

where $S = \int_0^1 \frac{\mathrm{Li}_2(t)^2}{t} dt$ and:

(i) $\int_0^{1/2} \frac{\mathrm{Li}_2(1-a)^2}{1-a} da = \int_{1/2}^1 \frac{\mathrm{Li}_2(r)^2}{r} dr = S - W$.

(ii) $W_3' := \int_0^{1/2} \frac{\mathrm{Li}_2(1-a)^2 - \zeta_2^2}{a} da = \int_{1/2}^1 \frac{\mathrm{Li}_2(r)^2 - \zeta_2^2}{1-r} dr$; integrating by parts ($v = -\ln(1-r)$; the boundary term at 1 vanishes since $\mathrm{Li}_2(r)^2 - \zeta_2^2 = O((1-r)\ln(1-r))$):

$$W_3' = -L(\mathrm{Li}_2(\tfrac{1}{2})^2 - \zeta_2^2) - 2\Big[\int_0^1 \frac{\ln^2(1-r)\mathrm{Li}_2(r)}{r} dr - V\Big] = -L(\mathrm{Li}_2(\tfrac{1}{2})^2 - \zeta_2^2) - 2(2\zeta_2\zeta_3 - \zeta_5) + 2V,$$

using $\int_0^1 \frac{\ln^2(1-r)\mathrm{Li}_2(r)}{r} dr = \sum_k \frac{H_k^2 + H_k^{(2)}}{k^3} = 2\zeta_2\zeta_3 - \zeta_5$ (the moment $\int_0^1 r^{k-1}\ln^2(1-r)\,dr = \frac{H_k^2 + H_k^{(2)}}{k}$, plus the classical Euler sums $\sum \frac{H_k^2}{k^3} = \frac{7}{2}\zeta_5 - \zeta_2\zeta_3$ and $\sum \frac{H_k^{(2)}}{k^3} = 3\zeta_2\zeta_3 - \frac{9}{2}\zeta_5$, both re-derived in Section 5.1).

(iii) $\alpha := \int_0^{1/2} \frac{\ln a\,(\zeta_3 - \mathrm{Li}_3(1-a))}{a} da = \frac{L^2}{2}(\zeta_3 - \mathrm{Li}_3(\frac{1}{2})) - \frac{1}{2}(2\zeta_2\zeta_3 - \zeta_5) + \frac{V}{2}$ (by parts with $\ln^2 a/2$, then $a \mapsto 1-a$ and the same full integral as in (ii)).

(iv) $\beta := \int_0^{1/2} \frac{\ln a\,(\zeta_3 - \mathrm{Li}_3(1-a))}{1-a} da = \zeta_3(\mathrm{Li}_2(\frac{1}{2}) - \zeta_2) - \zeta_2\zeta_3 + 3\zeta_5 - \mathrm{Li}_2(\frac{1}{2})\mathrm{Li}_3(\frac{1}{2}) + W$, using $r = 1-a$, $\int_0^1 \frac{\ln(1-r)\mathrm{Li}_3(r)}{r} dr = -\sum \frac{H_n}{n^4} = \zeta_2\zeta_3 - 3\zeta_5$ (Euler, §5.1), and Lemma 1.2 at $x = \frac{1}{2}$.

(v) $S = 2\zeta_2\zeta_3 - 3\zeta_5$: by parts, $S = \zeta_2\zeta_3 + \int_0^1 \frac{\ln(1-t)\mathrm{Li}_3(t)}{t} dt = \zeta_2\zeta_3 + (\zeta_2\zeta_3 - 3\zeta_5)$.

2.7 Summary. Combining (2.1), (2.4), (2.6):

$$\boxed{I_1 = -\zeta_2\zeta_3 + \zeta_2^2 L - L\,\mathfrak{T}_1 - \mathfrak{T}_2 - 2\,\mathfrak{T}_3 - \mathfrak{T}_4 + \Xi} \tag{2.7}$$

with (2.5)–(2.6). Every quantity on the right is an explicit constant, a $\mathrm{Li}_{s_1,\dots,s_k}(\frac{1}{2})$ of weight ≤ 5, or one of W, V. Identity (2.7) was verified numerically to 40 digits (each sub-identity separately as well).

3. The engine: all multiple polylogarithms at $\frac{1}{2}$ of weight ≤ 5

3.1 Words and values. For a word $w = a_1 a_2 \cdots a_n$ over $\{0,1\}$ (a_1 outermost) define

$$I(w) := \int_{0 < t_n < \cdots < t_1 < 1/2} \cdots \int \prod_{i=1}^n f_{a_i}(t_i)\,dt_i, \qquad f_0(t) = \frac{1}{t},\ f_1(t) = \frac{1}{1-t}.$$

$I(w)$ converges iff $a_n = 1$. Splitting w into blocks $0^{s_1-1}1\,0^{s_2-1}1\cdots$ gives $I(w) = \mathrm{Li}_{s_1,\dots,s_k}(\frac{1}{2})$ (expand each f_1 geometrically and integrate; all terms positive). There are $1, 2, 4, 8, 16$ convergent words of weights $1, \dots, 5$; $I(1) = L$.

Similarly, for words over $\{0, -1\}$ on $[0,1]$ with $f_{-1}(t) = \frac{1}{1+t}$, and more generally for signed compositions $\mathbf{c} = ((s_1, \sigma_1), \dots, (s_k, \sigma_k))$, $\sigma_i \in \{\pm 1\}$, define the **alternating Euler sums**

$$Z(\mathbf{c}) = \sum_{n_1 > \cdots > n_k \ge 1} \prod_i \frac{\sigma_i^{n_i}}{n_i^{s_i}},$$

convergent iff $(s_1, \sigma_1) \ne (1, +)$. The iterated-integral representation of $Z(\mathbf{c})$ uses the letters $y_i = \sigma_1 \cdots \sigma_i$: with $w(\mathbf{c}) = 0^{s_1-1}y_1 \cdots 0^{s_k-1}y_k$ one has $I_{[0,1]}(w(\mathbf{c})) = (-1)^k Z(\mathbf{c})$ (standard; verified numerically for all cases used).

3.2 Provable relation classes. The following families of linear relations hold, with elementary proofs:

(a) Shuffle. For convergent words u, v (on either interval), $I(u)I(v) = \sum_{w \in u \shuffle v} I(w)$: decompose the product of simplices into interleavings (Fubini on sets of full measure).

(b) Stuffle. For convergent signed compositions $\mathbf{u}, \mathbf{v}$: $Z(\mathbf{u})Z(\mathbf{v}) = \sum Z(\mathbf{t})$ over quasi-shuffles (interleavings with merges $(s, \sigma), (s', \sigma') \to (s + s', \sigma\sigma')$). *Proof:* the identity holds **exactly for partial sums cut at** N (finite rearrangement); let $N \to \infty$ (all terms convergent; conditionally convergent factors handled by Abel summation of the cutoff identity).

(c) Duplication. For an unsigned composition $\mathbf{s}$ with $s_1 \geq 2$, summing the even-index projection $\prod_i \frac{1+(-1)^{n_i}}{2}$: $\sum_{\boldsymbol{\sigma} \in \{\pm\}^k} Z(\mathbf{s}, \boldsymbol{\sigma}) = 2^{k-|\mathbf{s}|}\, \zeta(\mathbf{s})$.

(d) Regularized double shuffle (Hoffman relations). For every convergent $\mathbf{c}$,

$$\sum_{\substack{\mathbf{t} \in (1,+) * \mathbf{c} \\ \mathbf{t} \neq (1,+)\mathbf{c}}} Z(\mathbf{t}) = \sum_{\substack{w \in 1 \shuffle w(\mathbf{c}) \\ w \neq 1\, w(\mathbf{c})}} Z(w),$$

i.e. the stuffle and shuffle products with the divergent object "$\sum 1/m$" agree after removing the (identical) divergent leading term. *Proof.* Cut-off stuffle gives $H_N\, Z_{\leq N}(\mathbf{c}) = Z_{\leq N}((1,+)\mathbf{c}) + \sum_{\text{non-leading}} Z_{\leq N}(\mathbf{t})$ exactly. As $N \to \infty$, the left side is ζ-free of constant term: $H_N Z_{\leq N}(\mathbf{c}) = Z(\mathbf{c})H_N + o(1)$ (the tail of $Z(\mathbf{c})$ times $H_N \to 0$), while $Z_{\leq N}((1,+)\mathbf{c}) = Z(\mathbf{c})H_N - K_{\mathbf{c}} + o(1)$ with $K_{\mathbf{c}} = \sum_m \frac{Z(\mathbf{c}) - Z_{\leq m-1}(\mathbf{c})}{m} = \sum_n c_n H_n$ where c_n is the coefficient sequence of the outer index. Hence $\sum_{\text{nl}} Z(\mathbf{t}) = K_{\mathbf{c}}$. On the integral side, $-\ln(1-x)\, L_{\mathbf{c}}(x) = \sum_w L_w(x)$ exactly for $x < 1$ (shuffle with upper limit x), and as $x \to 1^-$ the same argument gives $\sum_{\text{nl}} Z(w) = K'_{\mathbf{c}}$ with $K'_{\mathbf{c}} = \int_0^1 \frac{Z(\mathbf{c}) - L_{\mathbf{c}}(t)}{1-t} dt = \sum_n c_n \sum_m \left(\frac{1}{m} - \frac{1}{m+n}\right) = \sum_n c_n H_n = K_{\mathbf{c}}$. $\square$

(e) Path composition (Chen) and reversal. For any weight-n word $x = 0 \cdots 1$ over $\{0, 1\}$ on $[0, 1]$ (an MZV word),

$$\zeta(x) = \sum_{k=0}^{n} I_{[\frac{1}{2},1]}(x_1 \cdots x_k)\, I(x_{k+1} \cdots x_n), \qquad I_{[\frac{1}{2},1]}(u) = I(\overline{u^R})$$

where $\overline{u^R}$ is u reversed with $0 \leftrightarrow 1$ (substitute $t \mapsto 1 - t$). All terms converge.

(f) Landen dictionary. The substitution $t = \frac{u}{1+u}$ maps $[0, \frac{1}{2}]$ to $[0, 1]$ with $\frac{dt}{t} = \frac{du}{u} - \frac{du}{1+u}$, $\frac{dt}{1-t} = \frac{du}{1+u}$. Hence every $I(w)$ (word at $\frac{1}{2}$) equals a signed sum of iterated integrals over $\{0, -1\}$ on $[0, 1]$, i.e. of alternating Euler sums:

$$I(w) = \sum_{\pm} \pm\, I_{[0,1]}(\text{words over } \{0, -1\}).$$

(g) Inputs. $Z((s, -)) = -\eta(s)$ with $\eta(s) = (1 - 2^{1-s})\zeta(s)$, $\eta(1) = L$ (elementary), and the unsigned MZVs of weight ≤ 5. The latter are *themselves consequences* of (a),(b),(d) plus **duality** ($t \mapsto 1 - t$ on $[0, 1]$: $\zeta(x) = \zeta(\overline{x^R})$): e.g. Hoffman(d) for $\mathbf{c} = (2)$ gives Euler's $\zeta(2, 1) = \zeta(3)$; at weight 4 one gets the sum theorem $\zeta(3, 1) + \zeta(2, 2) = \zeta_4$, and with the stuffle/shuffle squares of ζ_2 also $\zeta_4 = \frac{2}{5}\zeta_2^2$, $\zeta(2, 2) = \frac{3}{4}\zeta_4$, $\zeta(3, 1) = \frac{1}{4}\zeta_4$, $\zeta(2, 1, 1) = \zeta_4$; at weight 5, Hoffman(4) + stuffle/shuffle of $\zeta_2\zeta_3$ give

$$\zeta(4, 1) = 2\zeta_5 - \zeta_2\zeta_3, \quad \zeta(3, 2) = 3\zeta_2\zeta_3 - \tfrac{11}{2}\zeta_5, \quad \zeta(2, 3) = \tfrac{9}{2}\zeta_5 - 2\zeta_2\zeta_3,$$

and duality yields $\zeta(3, 1, 1) = \zeta(4, 1)$, $\zeta(2, 2, 1) = \zeta(3, 2)$, $\zeta(2, 1, 2) = \zeta(2, 3)$, $\zeta(2, 1, 1, 1) = \zeta_5$.

3.3 The linear system and its solution. Fix a weight $w \in \{2, 3, 4, 5\}$ and treat as unknowns *all* convergent alternating sums $Z(\mathbf{c})$ of weight w (there are $4, 12, 36, 108$) together with all $I(\text{word})$ at $\frac{1}{2}$ of weight w $(2, 4, 8, 16)$. Generate all relations (a)–(f) whose other ingredients have lower weight (hence are already known), plus the inputs (g). This yields an inhomogeneous linear system over $\mathbb{Q}$ with right-hand sides in the graded monomial space spanned at weight 5 by $\{\zeta_5, \zeta_2\zeta_3, \zeta_2^2 L, \zeta_3 L^2, \zeta_2 L^3, L^5, L\,\mathrm{Li}_4(\frac{1}{2}), \mathrm{Li}_5(\frac{1}{2})\}$.

Exact Gauss–Jordan elimination (performed by computer in rational arithmetic; the relation lists are machine-generated from the proved templates (a)–(g)) gives:

- weights 2 and 3: **every** unknown is determined. In particular the system *proves*

$$\mathrm{Li}_2(\tfrac{1}{2}) = \tfrac{\zeta_2}{2} - \tfrac{L^2}{2}, \qquad \mathrm{Li}_3(\tfrac{1}{2}) = \tfrac{7}{8}\zeta_3 - \tfrac{\zeta_2 L}{2} + \tfrac{L^3}{6}; \tag{3.9}$$

- weight 4: corank exactly 1; the single free direction is $\mathrm{Li}_4(\frac{1}{2})$ itself (as it must be — no closed form exists). All other values are determined in terms of it;
- weight 5: corank exactly 1, the free direction being $\mathrm{Li}_5(\frac{1}{2})$.

Every solved value was verified against direct numerical evaluation of the defining nested sums to ≥ 40 significant digits (all 30 half-words and spot checks across the 160 alternating sums).

Weight-4 table (all at argument $\frac{1}{2}$, $\mathrm{Li}_4 = \mathrm{Li}_4(\frac{1}{2})$):

$$\mathrm{Li}_{3,1} = \tfrac{L^4}{24} - \tfrac{L\zeta_3}{8} + \tfrac{\zeta_2^2}{20}, \quad \mathrm{Li}_{2,2} = \tfrac{L^4}{24} - \tfrac{L^2\zeta_2}{4} + \tfrac{L\zeta_3}{4} + \tfrac{\zeta_2^2}{40},$$
$$\mathrm{Li}_{2,1,1} = -\tfrac{L^4}{12} + \tfrac{L^2\zeta_2}{4} - \tfrac{7L\zeta_3}{8} - \mathrm{Li}_4 + \tfrac{2\zeta_2^2}{5}, \quad \mathrm{Li}_{1,3} = \tfrac{L^4}{24} - \tfrac{L^2\zeta_2}{4} + \tfrac{7L\zeta_3}{8} - \tfrac{\zeta_2^2}{8},$$
$$\mathrm{Li}_{1,2,1} = \tfrac{L^4}{12} - \tfrac{3L^2\zeta_2}{4} + \tfrac{11L\zeta_3}{4} + 3\,\mathrm{Li}_4 - \tfrac{6\zeta_2^2}{5},$$
$$\mathrm{Li}_{1,1,2} = -\tfrac{L^4}{6} + L^2\zeta_2 - \tfrac{23L\zeta_3}{8} - 3\,\mathrm{Li}_4 + \tfrac{6\zeta_2^2}{5}, \quad \mathrm{Li}_{1,1,1,1} = \tfrac{L^4}{24}.$$

(Equivalently $\sum \frac{H_n}{2^n n^3} = \mathrm{Li}_4 + \frac{\zeta_4}{8} - \frac{\zeta_3 L}{8} + \frac{L^4}{24}$ and $\sum \frac{H_n^{(2)}}{2^n n^2} = \mathrm{Li}_4 + \frac{\zeta_4}{16} + \frac{\zeta_3 L}{4} - \frac{\zeta_2 L^2}{4} + \frac{L^4}{24}$, the classical values.)

Weight-5 table (all at argument $\frac{1}{2}$; $\mathrm{Li}_4 = \mathrm{Li}_4(\frac{1}{2})$, $\mathrm{Li}_5 = \mathrm{Li}_5(\frac{1}{2})$, and $\zeta_4 = \frac{2}{5}\zeta_2^2$):

value	closed form
$\mathrm{Li}_{4,1}$	$\mathrm{Li}_5 + L\,\mathrm{Li}_4 + \frac{\zeta_5}{32} - \frac{\zeta_2\zeta_3}{2} + \frac{L^2\zeta_3}{2} - \frac{L\zeta_2^2}{20} - \frac{L^3\zeta_2}{6} + \frac{L^5}{40}$
$\mathrm{Li}_{3,2}$	$-3\,\mathrm{Li}_5 - 3L\,\mathrm{Li}_4 + \frac{23\zeta_5}{64} + \frac{23\zeta_2\zeta_3}{16} - \frac{23L^2\zeta_3}{16} - \frac{L\zeta_2^2}{40} + \frac{7L^3\zeta_2}{12} - \frac{13L^5}{120}$
$\mathrm{Li}_{3,1,1}$	$-\mathrm{Li}_5 + \frac{63\zeta_5}{32} - \frac{\zeta_2\zeta_3}{2} + \frac{7L^2\zeta_3}{16} - \frac{2L\zeta_2^2}{5} - \frac{L^3\zeta_2}{12} + \frac{L^5}{60}$
$\mathrm{Li}_{2,3}$	$3\,\mathrm{Li}_5 + 3L\,\mathrm{Li}_4 - \frac{81\zeta_5}{64} - \frac{7\zeta_2\zeta_3}{8} + \frac{7L^2\zeta_3}{8} + \frac{L\zeta_2^2}{8} - \frac{5L^3\zeta_2}{12} + \frac{11L^5}{120}$
$\mathrm{Li}_{2,2,1}$	$3\,\mathrm{Li}_5 - \frac{375\zeta_5}{64} + \frac{3\zeta_2\zeta_3}{2} - \frac{11L^2\zeta_3}{8} + \frac{6L\zeta_2^2}{5} + \frac{L^3\zeta_2}{4} - \frac{L^5}{60}$
$\mathrm{Li}_{2,1,2}$	$-3\,\mathrm{Li}_5 + \frac{369\zeta_5}{64} - \frac{23\zeta_2\zeta_3}{16} + \frac{23L^2\zeta_3}{16} - \frac{6L\zeta_2^2}{5} - \frac{L^3\zeta_2}{3} + \frac{L^5}{30}$
$\mathrm{Li}_{2,1,1,1}$	$-\mathrm{Li}_5 - L\,\mathrm{Li}_4 + \zeta_5 - \frac{7L^2\zeta_3}{16} + \frac{L^3\zeta_2}{6} - \frac{L^5}{24}$
$\mathrm{Li}_{1,4}$	$-2\,\mathrm{Li}_5 - L\,\mathrm{Li}_4 + \frac{27\zeta_5}{32} + \frac{7\zeta_2\zeta_3}{16} - \frac{7L^2\zeta_3}{16} + \frac{L^3\zeta_2}{6} - \frac{L^5}{30}$
$\mathrm{Li}_{1,3,1}$	$-\frac{3\zeta_5}{64} + \frac{L\zeta_2^2}{20} - \frac{L^2\zeta_3}{16} + \frac{L^5}{120}$
$\mathrm{Li}_{1,2,2}$	$\frac{3\zeta_5}{16} - \frac{\zeta_2\zeta_3}{8} + \frac{L^2\zeta_3}{8} + \frac{L\zeta_2^2}{40} - \frac{L^3\zeta_2}{12} + \frac{L^5}{120}$
$\mathrm{Li}_{1,2,1,1}$	$4\,\mathrm{Li}_5 + 3L\,\mathrm{Li}_4 - 4\zeta_5 + \frac{7L^2\zeta_3}{8} + \frac{2L\zeta_2^2}{5} - \frac{5L^3\zeta_2}{12} + \frac{L^5}{12}$
$\mathrm{Li}_{1,1,3}$	$-\frac{3\zeta_5}{64} + \frac{\zeta_2\zeta_3}{16} + \frac{7L^2\zeta_3}{16} - \frac{L\zeta_2^2}{8} - \frac{L^3\zeta_2}{12} + \frac{L^5}{120}$
$\mathrm{Li}_{1,1,2,1}$	$-6\,\mathrm{Li}_5 - 3L\,\mathrm{Li}_4 + 6\zeta_5 + \frac{L^2\zeta_3}{16} - \frac{6L\zeta_2^2}{5} + \frac{L^3\zeta_2}{4} - \frac{L^5}{12}$
$\mathrm{Li}_{1,1,1,2}$	$4\,\mathrm{Li}_5 + L\,\mathrm{Li}_4 - 4\zeta_5 - L^2\zeta_3 + \frac{6L\zeta_2^2}{5} + \frac{L^3\zeta_2}{6}$
$\mathrm{Li}_{1,1,1,1,1}$	$\frac{L^5}{120}$

3.4 The four core constants. From the table, using $X = \mathrm{Li}_{4,1} + \mathrm{Li}_5$, $Y = \mathrm{Li}_{3,2} + \mathrm{Li}_5$, $W = 2\,\mathrm{Li}_{3,2} + 4\,\mathrm{Li}_{4,1}$ (shuffle $\mathrm{Li}_2(t)^2 = 2\,\mathrm{Li}_{2,2}(t) + 4\,\mathrm{Li}_{3,1}(t)$ integrated against dt/t), and $V = 2\sum_{w\in 11\,\Pi\,01} I(0w)$:

$$X = \sum \frac{H_n}{2^n n^4} = 2\,\mathrm{Li}_5 + L\,\mathrm{Li}_4 + \frac{\zeta_5}{32} - \frac{\zeta_2\zeta_3}{2} + \frac{L^2\zeta_3}{2} - \frac{L\zeta_2^2}{20} - \frac{L^3\zeta_2}{6} + \frac{L^5}{40}, \tag{3.10}$$

$$Y=\sum\frac{H_n^{(2)}}{2^n n^3}=-2\,\mathrm{Li}_5-3L\,\mathrm{Li}_4+\frac{23\zeta_5}{64}+\frac{23\zeta_2\zeta_3}{16}-\frac{23L^2\zeta_3}{16}-\frac{L\zeta_2^2}{40}+\frac{7L^3\zeta_2}{12}-\frac{13L^5}{120},\tag{3.11}$$

$$W=-2\,\mathrm{Li}_5-2L\,\mathrm{Li}_4+\frac{27\zeta_5}{32}+\frac{7\zeta_2\zeta_3}{8}-\frac{7L^2\zeta_3}{8}-\frac{L\zeta_2^2}{4}+\frac{L^3\zeta_2}{2}-\frac{7L^5}{60},\tag{3.12}$$

$$V=-\frac{3\zeta_5}{32}+\frac{\zeta_2\zeta_3}{8}-\frac{L^3\zeta_2}{6}+\frac{L^5}{10}.\tag{3.13}$$

((3.12), (3.13) verified against direct quadrature of the defining integrals to 40 digits; (0.9) holds identically.)

4. Assembly

I_2**:** insert (0.7) and (3.12) into (1.2). All $\mathrm{Li}_4, \mathrm{Li}_5$ contributions combine to $2L\,\mathrm{Li}_4(\tfrac{1}{2})+2\,\mathrm{Li}_5(\tfrac{1}{2})$, giving the boxed result (with $\zeta_2^2=\tfrac{5}{2}\zeta_4$).

I_1**:** insert the weight-5 table into (2.5), then (2.5)–(2.6) and (3.12), (3.13) into (2.7). Exact rational arithmetic gives

$$I_1=\frac{29}{16}\zeta_5-\frac{19}{16}\zeta_2\zeta_3+\frac{11}{40}\zeta_2^2L+\frac{5}{16}\zeta_3L^2-\frac{5}{12}\zeta_2L^3+\frac{L^5}{40}-3\,\mathrm{Li}_5(\tfrac{1}{2}),$$

i.e. the boxed result ($\tfrac{11}{40}\zeta_2^2=\tfrac{11}{16}\zeta_4$). The $\mathrm{Li}_4(\tfrac{1}{2})$-terms cancel identically.

5. Independent cross-checks

5.1 Classical Euler sums used in §2.6 (also outputs of the engine): the symmetric ("partial fraction") method. From $\frac{1}{m^p(m+k)^q}=\sum_{i=1}^{p}\binom{p+q-i-1}{q-1}\frac{(-1)^{p-i}}{k^{p+q-i}m^i}+\sum_{j=1}^{q}\binom{p+q-j-1}{p-1}\frac{(-1)^p}{k^{p+q-j}(m+k)^j}$ (a polynomial identity, checked symbolically) one derives in the standard way $\sum\frac{H_n}{n^4}=3\zeta_5-\zeta_2\zeta_3$, $\sum\frac{H_n^{(2)}}{n^3}=3\zeta_2\zeta_3-\frac{9}{2}\zeta_5$; and $\sum\frac{H_n^2}{n^3}=\frac{7}{2}\zeta_5-\zeta_2\zeta_3$ follows from $H_n^2=H_n^{(2)}+2\sum_{i<j\le n}\frac{1}{ij}$, the duality $\zeta(3,1,1)=\zeta(4,1)$ and $\zeta(4,1)=2\zeta_5-\zeta_2\zeta_3$.

5.2 Alternating linear sums. Applying the same partial-fraction identity to the doubly-signed double sums $A(p,q)=\sum(-1)^{n+1}H_n^{(p)}/n^q$, $D'(p,q)=\sum_{m<n}(-1)^{n+m}/(n^qm^p)$, $E'(p,q)=\sum_{m<n}(-1)^{m+1}/(m^pn^q)$ (with the H_k-telescope grouping of the divergent pieces) together with the $\eta\cdot\eta$ and $\zeta\cdot\eta$ stuffles yields a 12×16 linear system whose unique solution includes

$$A(1,4)=\tfrac{59}{32}\zeta_5-\tfrac{1}{2}\zeta_2\zeta_3,\qquad A(2,3)=\tfrac{5}{8}\zeta_2\zeta_3-\tfrac{11}{32}\zeta_5,$$

in agreement with the engine's output — an independent re-derivation by a different mechanism.

5.3 An independent integral-equation system for X. The four relations (i) $W=2Y+4X-6\mathrm{Li}_5(\tfrac{1}{2})$ (0.9); (ii) the Landen substitution $t=y/(1+y)$ applied to W, which after expanding $\mathrm{Li}_2(t)=-\mathrm{Li}_2(-y)-\tfrac{1}{2}\ln^2(1+y)$ reduces W to $\int_0^1\frac{\mathrm{Li}_2(-y)^2}{y}dy=-2A(2,3)-4A(1,4)+6\eta(5)$, $\int_0^1\frac{\ln^4(1+y)}{y}dy$ (elementary) and V; (iii) the reflection split of S; and (iv) the reflection split of $\int_0^1\frac{\ln^2(1-u)\mathrm{Li}_2(u)}{u}du=2\zeta_2\zeta_3-\zeta_5$, form a rank-3 system in (X,Y,W,V) which determines X — and the value agrees with (3.10). Likewise a weight-4 version determines $\sum\frac{H_n}{2^nn^3}$, $\sum\frac{H_n^{(2)}}{2^nn^2}$ and $\sum\frac{(-1)^{n-1}H_n}{n^3}=\frac{11}{4}\zeta_4-2\mathrm{Li}_4(\tfrac{1}{2})-\frac{7}{4}\zeta_3L+\frac{1}{2}\zeta_2L^2-\frac{1}{12}L^4$, agreeing with §3.3.

5.4 PSLQ. Independently of all of the above, PSLQ at 100 decimal digits applied to (I_1,I_2) against the 8-element weight-5 basis recovers exactly the boxed coefficient vectors.

Numerical verification

Direct high-precision quadrature (mpmath, tanh–sinh, dps = 120, after the substitution $u = 1 - x$ which removes the endpoint singularity):

```
I1 (quadrature) = -1.52182292885581001525839248485539311533028568041866530576195346496728998166708957154751890942661667659069
I1 (closed form) = identical to all 105 displayed digits
I2 (quadrature) =  3.32096687866486768034598612120489450967715868430766556485285797301857262754460209508907350506879612686912
I2 (closed form) = identical to all 105 displayed digits
```

Agreement: ≥ 100 **significant digits** for both integrals ($|I_1 - C_1| < 10^{-120}$, $|I_2 - C_2| < 10^{-120}$). In addition: every lemma and reduction step in §§1–2 was verified numerically to 26–70 digits; all 30 multiple-polylog values at $\frac{1}{2}$ of §3.3 were verified to ≥ 40 digits against their defining sums; the alternating-sum cross-checks of §5.2 were verified to 50 digits.

Notes

1. **Completeness of the derivation.** Every reduction step (Sections 1, 2, 4) is proved in full. The evaluation engine (Section 3) rests on the six relation classes (a)–(f) plus the inputs (g); each class is proved elementarily in the text (shuffle = Fubini on simplices; stuffle = finite cutoff identity + Abel; duplication = parity projection; Hoffman = the $K_{\mathbf{c}} = \sum_n c_n H_n$ computation given in (d); path composition = domain decomposition; reversal and Landen = substitutions $t \mapsto 1 - t$, $t \mapsto u/(1+u)$). The *generation* of the several hundred instances of these relations and the exact rational Gauss–Jordan elimination were carried out by computer; this is bookkeeping, not new mathematics, and the resulting solved values were verified numerically to ≥ 40 digits each, with the coranks coming out exactly $\{0, 0, 1, 1\}$ at weights $2, 3, 4, 5$ (free directions = $\mathrm{Li}_4(\frac{1}{2}), \mathrm{Li}_5(\frac{1}{2})$), as the known dimension of the level-2 weight-graded space predicts.

2. The two integrals are "level 2, weight 5" and cannot be expressed in ζ-values and $\ln 2$ alone; $\mathrm{Li}_4(\frac{1}{2})$ and $\mathrm{Li}_5(\frac{1}{2})$ are the standard irreducibles. The disappearance of $\mathrm{Li}_4(\frac{1}{2})$ from I_1 is a genuine cancellation.

3. The same machinery yields, as by-products, closed forms for all $\sum H_n^{(p)}/(2^n n^q)$ of weight ≤ 5 (values (3.10)–(3.13) and the tables), reproducing the classical results of the literature on Euler sums at $\frac{1}{2}$.