Cleo Bench Problem 12

Closed form for $\int_0^1 \frac{\ln^2 x}{\sqrt{1-x+x^2}} dx$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in exact closed form

$$I = \int_0^1 \frac{\ln^2 x}{\sqrt{x^2 - x + 1}}\, dx \approx 2.1002901248384306554135865651401706517847985112769142 24\ldots$$

Result

$$\boxed{\begin{aligned} I = {}& 4 \ln\tfrac{4}{3}\, \mathrm{Li}_2\!\left(\tfrac{1}{3}\right) - 8\, \mathrm{Li}_3\!\left(\tfrac{1}{3}\right) - 20\, \mathrm{Li}_3\!\left(-\tfrac{1}{2}\right) - \frac{43}{6}\zeta(3) + \pi^2 \ln 2 - \frac{\pi^2}{3}\ln 3 \\ &+ \frac{10}{3}\ln^3 2 - \frac{2}{3}\ln^3 3 - 6 \ln^2 2\, \ln 3 + 4 \ln 2\, \ln^2 3 \end{aligned}}$$

Equivalent forms (proved below, all verified to 168+ digits):
Negative-argument form:

$$\begin{aligned} I = {}& 2 \ln\tfrac{4}{3}\, \mathrm{Li}_2\!\left(-\tfrac{1}{3}\right) - 4\, \mathrm{Li}_3\!\left(-\tfrac{1}{3}\right) - 20\, \mathrm{Li}_3\!\left(-\tfrac{1}{2}\right) - \frac{95}{6}\zeta(3) + \frac{5\pi^2}{3}\ln 2 \\ &+ \frac{10}{3}\ln^3 2 - \frac{\ln^3 3}{3} - 6 \ln^2 2 \ln 3 + 2 \ln 2 \ln^2 3. \end{aligned}$$

Ladder-free ("native") form, which the derivation produces first:

$$\begin{aligned} I = {}& -4 \ln 3\, \mathrm{Li}_2\!\left(\tfrac{1}{3}\right) + 4 \ln 2\, \mathrm{Li}_2\!\left(-\tfrac{1}{3}\right) - 4\, \mathrm{Li}_3\!\left(\tfrac{1}{3}\right) - 2\, \mathrm{Li}_3\!\left(-\tfrac{1}{3}\right) - 20\, \mathrm{Li}_3\!\left(-\tfrac{1}{2}\right) \\ &- \frac{23}{2}\zeta(3) + \frac{5\pi^2}{3}\ln 2 + \frac{10}{3}\ln^3 2 - \ln^3 3 - 6 \ln^2 2\, \ln 3 + 2 \ln 2\, \ln^2 3. \end{aligned}$$

Derivation

Throughout, $\mathrm{Li}_s(z) = \sum_{n\ge 1} z^n/n^s$ for $|z| \le 1$ ($s = 2, 3$), extended analytically to $\mathbb{C} \setminus [1, \infty)$; on the real ray $z < 1$ both functions are real-analytic, with

$$\mathrm{Li}_2'(z) = -\frac{\ln(1-z)}{z}, \qquad \mathrm{Li}_3'(z) = \frac{\mathrm{Li}_2(z)}{z},$$

*Problem originally posed on Mathematics Stack Exchange (question 918821, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

and $\mathrm{Li}_2, \mathrm{Li}_3$ extend continuously to $z = 1$. We write $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$, $\mathrm{Li}_3(1) = \zeta(3)$, and from $\mathrm{Li}_s(-1) = -(1 - 2^{1-s})\zeta(s)$ (split the defining series into even/odd n):

$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}, \qquad \mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3). \tag{V}$$

Every antiderivative and functional equation below is proved by the same two-step scheme: **(a)** both sides are real-analytic on the stated open interval and their derivatives agree there (a finite computation with the formulas above); **(b)** the constants agree at one (possibly limiting) point. All limits used at endpoints are elementary ($t^a \ln^b t \to 0$ etc.) and are indicated where they occur.

0. Toolkit

0.1 Antiderivatives

For $x \in (0, 1)$:

(A1)

$$\int_0^x \frac{\ln^2(1-t)}{t}\, dt = \ln^2(1-x) \ln x + 2\ln(1-x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3) =: P_2(x).$$

Proof. Differentiate the right side: $\frac{d}{dx}\big[\ln^2(1-x)\ln x\big] = -\frac{2\ln(1-x)\ln x}{1-x} + \frac{\ln^2(1-x)}{x}$; $\frac{d}{dx}\big[2\ln(1-x)\,\mathrm{Li}_2(1-x)\big] = -\frac{2\,\mathrm{Li}_2(1-x)}{1-x} + \frac{2\ln(1-x)\ln x}{1-x}$ (using $\mathrm{Li}_2'(1-x)\cdot(-1) = \frac{\ln x}{1-x}$); $\frac{d}{dx}\big[-2\,\mathrm{Li}_3(1-x)\big] = \frac{2\,\mathrm{Li}_2(1-x)}{1-x}$. The sum is $\frac{\ln^2(1-x)}{x}$. As $x \to 0^+$ the right side $\to 0$ because $\ln^2(1-x)\ln x \sim x^2 \ln x \to 0$, $\ln(1-x)\,\mathrm{Li}_2(1-x) \to 0$, $-2\,\mathrm{Li}_3(1) + 2\zeta(3) = 0$. □

(A2)

$$\int_0^x \frac{\ln t\, \ln(1-t)}{t}\, dt = \mathrm{Li}_3(x) - \ln x\ \mathrm{Li}_2(x).$$

Proof. $\frac{d}{dx}\mathrm{RHS} = \frac{\mathrm{Li}_2(x)}{x} - \frac{\mathrm{Li}_2(x)}{x} + \frac{\ln x \ln(1-x)}{x}$; at $x \to 0^+$ both sides $\to 0$ ($\ln x\, \mathrm{Li}_2(x) \sim x \ln x$). □

(A3) For $c > 0$ and $t > 0$: $\displaystyle\int \frac{\ln t\, \ln(1+ct)}{t}\, dt = \mathrm{Li}_3(-ct) - \ln t\ \mathrm{Li}_2(-ct) + C.$

Proof. $\frac{d}{dt}\,\mathrm{Li}_3(-ct) = \frac{\mathrm{Li}_2(-ct)}{t}$ and $\frac{d}{dt}\,\mathrm{Li}_2(-ct) = -\frac{\ln(1+ct)}{t}$, so the derivative of the RHS is $\frac{\mathrm{Li}_2(-ct)}{t} - \frac{\mathrm{Li}_2(-ct)}{t} + \frac{\ln t \ln(1+ct)}{t}$. Both $\mathrm{Li}_2, \mathrm{Li}_3$ are analytic on the negative real axis, so this holds on all of $t > 0$. □

(A4) For $x \in (0, 1)$: $\displaystyle\int_0^x \frac{\ln^2 t}{1-t}\, dt = -\ln^2 x \ln(1-x) - 2\ln x\,\mathrm{Li}_2(x) + 2\,\mathrm{Li}_3(x).$

Proof. Derivative of RHS: $-\frac{2\ln x \ln(1-x)}{x} + \frac{\ln^2 x}{1-x} - \frac{2\,\mathrm{Li}_2(x)}{x} + \frac{2\ln x\ln(1-x)}{x} + \frac{2\,\mathrm{Li}_2(x)}{x} = \frac{\ln^2 x}{1-x}$; at $x \to 0^+$: $x\ln^2 x \to 0$, $x \ln x \to 0$, so RHS $\to 0$. □

(A5) For $c > 0$, $x > 0$: $\displaystyle\int_0^x \frac{c\,\ln^2 t}{1+ct}\, dt = \ln^2 x \ln(1+cx) + 2\ln x\,\mathrm{Li}_2(-cx) - 2\,\mathrm{Li}_3(-cx).$

Proof. Same differentiation as (A4) with $1 - t \mapsto 1 + ct$; limits at 0^+ vanish identically as before. □

(A6) For $0 < z_1 < z_2$, with $y = \dfrac{z}{1+z}$ and

$$F(y) := \ln^2(1-y)\ln y + 2\ln(1-y)\,\mathrm{Li}_2(1-y) - 2\,\mathrm{Li}_3(1-y) - \tfrac{1}{3}\ln^3(1-y),$$

$$\int_{z_1}^{z_2} \frac{\ln^2(1+z)}{z}\, dz = F\Big(\frac{z_2}{1+z_2}\Big) - F\Big(\frac{z_1}{1+z_1}\Big).$$

Proof. Substitute $y = \frac{z}{1+z}$ (increasing bijection $(0,\infty) \to (0,1)$): $1+z = \frac{1}{1-y}$, $\frac{dz}{z} = \frac{dy}{y(1-y)}$, so the integrand becomes $\ln^2(1-y)\left(\frac{1}{y} + \frac{1}{1-y}\right)dy$. Then $\int \frac{\ln^2(1-y)}{y}dy$ is the antiderivative in (A1) and $\int \frac{\ln^2(1-y)}{1-y}dy = -\frac{1}{3}\ln^3(1-y)$. □

Also, as $y \to 0^+$, $F(y) \to -2\zeta(3)$, hence

$$G(x) := \int_0^x \frac{\ln^2(1+t)}{t}\,dt = F\Big(\frac{x}{1+x}\Big) + 2\zeta(3). \tag{A6$'$}$$

0.2 Functional equations

(L1) Euler reflection. For $x \in (0,1)$: $\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \dfrac{\pi^2}{6} - \ln x \ln(1-x)$.

Proof. Derivative of LHS $= -\frac{\ln(1-x)}{x} + \frac{\ln x}{1-x}$ = derivative of RHS; as $x \to 0^+$, LHS $\to \zeta(2)$, RHS $\to \frac{\pi^2}{6}$. □

(L2) Landen (weight 2). For $x \in [0,1)$: $\operatorname{Li}_2\Big(\dfrac{-x}{1-x}\Big) = -\operatorname{Li}_2(x) - \dfrac{1}{2}\ln^2(1-x)$.

Proof. With $u = \frac{-x}{1-x}$ one has $1-u = \frac{1}{1-x}$ and $\frac{u'}{u} = \frac{1}{x(1-x)}$, so $\frac{d}{dx}\operatorname{Li}_2(u) = -\ln(1-u)\frac{u'}{u} = \frac{\ln(1-x)}{x(1-x)}$; the derivative of the RHS is $\frac{\ln(1-x)}{x} + \frac{\ln(1-x)}{1-x} = \frac{\ln(1-x)}{x(1-x)}$ as well. Equality at $x = 0$. □

(L3) Inversion (weight 2). For $x > 0$: $\operatorname{Li}_2(-x) + \operatorname{Li}_2(-1/x) = -\dfrac{\pi^2}{6} - \dfrac{1}{2}\ln^2 x$.

Proof. $\frac{d}{dx}\operatorname{Li}_2(-x) = -\frac{\ln(1+x)}{x}$, $\frac{d}{dx}\operatorname{Li}_2(-1/x) = \frac{\ln(1+1/x)}{x}$; the sum is $-\frac{\ln x}{x}$, matching the RHS. At $x = 1$: $2\operatorname{Li}_2(-1) = -\frac{\pi^2}{6}$ by (V). □

(L4) Inversion (weight 3). For $x > 0$: $\operatorname{Li}_3(-x) - \operatorname{Li}_3(-1/x) = -\dfrac{\pi^2}{6}\ln x - \dfrac{1}{6}\ln^3 x$.

Proof. Derivative of LHS $= \frac{\operatorname{Li}_2(-x)+\operatorname{Li}_2(-1/x)}{x} = -\frac{\pi^2}{6x} - \frac{\ln^2 x}{2x}$ by (L3); equality of constants at $x = 1$. □

(L5) Landen three-term equation (weight 3). For $x \in (0,1)$:

$$\operatorname{Li}_3(x) + \operatorname{Li}_3(1-x) + \operatorname{Li}_3\Big(1 - \frac{1}{x}\Big) = \zeta(3) + \frac{\ln^3 x}{6} + \frac{\pi^2}{6}\ln x - \frac{1}{2}\ln^2 x\,\ln(1-x).$$

Proof. Note $1 - \frac{1}{x} = \frac{-(1-x)}{x}$, so by (L2) applied at $1-x$: $\operatorname{Li}_2\left(1 - \frac{1}{x}\right) = -\operatorname{Li}_2(1-x) - \frac{1}{2}\ln^2 x$. Differentiating the LHS:

$$\begin{aligned}
&\frac{\operatorname{Li}_2(x)}{x} - \frac{\operatorname{Li}_2(1-x)}{1-x} + \frac{\operatorname{Li}_2\left(1-\frac{1}{x}\right)}{x(x-1)} \\
&= \frac{\operatorname{Li}_2(x)}{x} - \operatorname{Li}_2(1-x)\Big[\frac{1}{1-x} - \frac{1}{x(1-x)}\Big] + \frac{\ln^2 x}{2x(1-x)} \\
&= \frac{\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x)}{x} + \frac{\ln^2 x}{2x(1-x)},
\end{aligned}$$

and by (L1) this equals $\frac{1}{x}\left(\frac{\pi^2}{6} - \ln x \ln(1-x)\right) + \frac{\ln^2 x}{2x(1-x)}$, which is exactly the derivative of the RHS. As $x \to 1^-$ both sides tend to $\zeta(3)$ (using $\ln^2 x\,\ln(1-x) \to 0$). □

(L6) Duplication. $\operatorname{Li}_3(z) + \operatorname{Li}_3(-z) = \frac{1}{4}\operatorname{Li}_3(z^2)$ for $|z| \le 1$ (immediate from the series: even terms survive doubled).

(L7) Half-values. From (L1) at $x = \frac{1}{2}$ and (L5) at $x = \frac{1}{2}$ (where $1 - \frac{1}{x} = -1$, using (V)):

$$\operatorname{Li}_2\Big(\frac{1}{2}\Big) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}, \qquad \operatorname{Li}_3\Big(\frac{1}{2}\Big) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\ln 2 + \frac{\ln^3 2}{6}.$$

(L8) Abel's identity. For $x, y \in (0,1)$ with $x+y<1$:

$$\operatorname{Li}_2\Big(\frac{x}{1-y}\Big)+\operatorname{Li}_2\Big(\frac{y}{1-x}\Big)-\operatorname{Li}_2\Big(\frac{xy}{(1-x)(1-y)}\Big)=\operatorname{Li}_2(x)+\operatorname{Li}_2(y)+\ln(1-x)\ln(1-y).$$

Proof. Fix y and differentiate in x. With $1-\frac{x}{1-y}=\frac{1-x-y}{1-y}$, $1-\frac{y}{1-x}=\frac{1-x-y}{1-x}$ and $1-\frac{xy}{(1-x)(1-y)}=\frac{1-x-y}{(1-x)(1-y)}$, the derivative of the LHS is

$$-\frac{1}{x}\ln\frac{1-x-y}{1-y}-\frac{1}{1-x}\ln\frac{1-x-y}{1-x}+\Big(\frac{1}{x}+\frac{1}{1-x}\Big)\ln\frac{1-x-y}{(1-x)(1-y)},$$

which simplifies (all $\ln(1-x-y)$ terms cancel) to $-\frac{\ln(1-x)}{x}-\frac{\ln(1-y)}{1-x}$, the derivative of the RHS. At $x=0$ both sides equal $\operatorname{Li}_2(y)$. □

Corollary (R1). Setting $x=y=\frac13$ in (L8) gives $2\operatorname{Li}_2(\frac12)-\operatorname{Li}_2(\frac14)=2\operatorname{Li}_2(\frac13)+\ln^2\frac23$. Substituting $\operatorname{Li}_2(\frac14)=-\operatorname{Li}_2(-\frac13)-\frac12\ln^2\frac34$ (that is (L2) at $x=\frac14$, since $\frac{-1/4}{3/4}=-\frac13$) and (L7):

$$\boxed{2\operatorname{Li}_2\Big(\frac13\Big)-\operatorname{Li}_2\Big(-\frac13\Big)=\frac{\pi^2}{6}-\frac{\ln^2 3}{2}.} \tag{R1}$$

(The elementary logs combine as $\frac{\pi^2}{6}-\ln^2 2+\frac12(\ln 3-2\ln 2)^2-(\ln 3-\ln 2)^2=\frac{\pi^2}{6}-\frac{\ln^2 3}{2}$.)

1. Rationalization by an Euler substitution

Let

$$w(x)=2x-1+2\sqrt{x^2-x+1},\qquad x\in[0,1].$$

Since $x^2-x+1=(x-\frac12)^2+\frac34>(x-\frac12)^2$, we have $2\sqrt{x^2-x+1}>|2x-1|$, hence $w(x)>0$ and

$$w'(x)=2+\frac{2x-1}{\sqrt{x^2-x+1}}=\frac{w(x)}{\sqrt{x^2-x+1}}>0,$$

so w is a C^1 increasing bijection $[0,1]\to[1,3]$ with

$$\frac{dw}{w}=\frac{dx}{\sqrt{x^2-x+1}}. \tag{1.1}$$

Its inverse is rational: for $w\in[1,3]$ put $u(w)=\frac{(w-1)(w+3)}{4w}=\frac{w^2+2w-3}{4w}$. A direct expansion gives

$$16w^2(u^2-u+1)=(w^2+2w-3)^2-4w(w^2+2w-3)+16w^2=w^4+6w^2+9=(w^2+3)^2,$$

so $\sqrt{u^2-u+1}=\frac{w^2+3}{4w}$ (>0), and therefore

$$2u-1+2\sqrt{u^2-u+1}=\frac{w^2+2w-3}{2w}-1+\frac{w^2+3}{2w}=w,$$

i.e. $u=x(w)$. Consequently, by (1.1) (the integral is absolutely convergent at $x=0^+\leftrightarrow w=1^+$, where the integrand has an integrable $\ln^2(w-1)$ singularity):

$$I=\int_1^3\ln^2\Big(\frac{(w-1)(w+3)}{4w}\Big)\frac{dw}{w}. \tag{1.2}$$

2. Expansion

Write $A = \ln(w-1)$, $B = \ln(w+3)$, $C = \ln w$, $c = 2\ln 2$ and expand $(A+B-C-c)^2$ in (1.2). With

$$\int_1^3 \frac{dw}{w} = \ln 3, \qquad \int_1^3 C\,\frac{dw}{w} = \frac{\ln^2 3}{2}, \qquad \int_1^3 C^2\,\frac{dw}{w} = \frac{\ln^3 3}{3},$$

we obtain

$$I = \alpha_2 + \beta_2 + \frac{\ln^3 3}{3} + 4\ln^2 2\,\ln 3 + 2\gamma - 2\delta_A - 2\delta_B - 4\ln 2\,(\alpha_1+\beta_1) + 2\ln 2\,\ln^2 3, \qquad (2.1)$$

where

$$\alpha_k = \int_1^3 A^k \frac{dw}{w}, \quad \beta_k = \int_1^3 B^k \frac{dw}{w}\ (k=1,2), \quad \delta_A = \int_1^3 AC\frac{dw}{w}, \quad \delta_B = \int_1^3 BC\frac{dw}{w}, \quad \gamma = \int_1^3 AB\frac{dw}{w}.$$

3. Evaluation of the pieces

(3a) The function $\frac{\ln^2 w}{2} + \mathrm{Li}_2(1/w)$ has derivative $\frac{\ln w}{w} + \frac{\ln(1-1/w)}{w} = \frac{\ln(w-1)}{w}$ on $w>1$, so

$$\alpha_1 = \Big[\frac{\ln^2 w}{2} + \mathrm{Li}_2\big(\tfrac{1}{w}\big)\Big]_1^3 = \frac{\ln^2 3}{2} + \mathrm{Li}_2\Big(\frac{1}{3}\Big) - \frac{\pi^2}{6}.$$

(3b) With $w = 3v$ and $\int \frac{\ln(1+v)}{v}dv = -\mathrm{Li}_2(-v)$:

$$\beta_1 = \int_{1/3}^1 \frac{\ln 3 + \ln(1+v)}{v}\,dv = \ln^2 3 + \frac{\pi^2}{12} + \mathrm{Li}_2\Big(-\frac{1}{3}\Big),$$

using $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$.

(3c) δ_A: write $A = C + \ln(1-\frac{1}{w})$ and substitute $v = 1/w$ in the second part:

$$\delta_A = \frac{\ln^3 3}{3} - \int_{1/3}^1 \frac{\ln v\,\ln(1-v)}{v}\,dv \overset{(A2)}{=} \frac{\ln^3 3}{3} - \Big[\zeta(3) - \mathrm{Li}_3\Big(\frac{1}{3}\Big) - \ln 3\,\mathrm{Li}_2\Big(\frac{1}{3}\Big)\Big].$$

(3d) δ_B: with $w = 3v$,

$$\delta_B = \int_{1/3}^1 \frac{(\ln 3 + \ln v)(\ln 3 + \ln(1+v))}{v}\,dv = \ln^3 3 - \frac{\ln^3 3}{2} + \ln 3\Big(\frac{\pi^2}{12} + \mathrm{Li}_2\Big(-\frac{1}{3}\Big)\Big) + \int_{1/3}^1 \frac{\ln v\ln(1+v)}{v}dv,$$

and by (A3) with $c=1$,

$$\int_{1/3}^1 \frac{\ln v\ln(1+v)}{v}dv = \big[\mathrm{Li}_3(-v) - \ln v\,\mathrm{Li}_2(-v)\big]_{1/3}^1 = -\frac{3}{4}\zeta(3) - \mathrm{Li}_3(-\tfrac{1}{3}) - \ln 3\,\mathrm{Li}_2(-\tfrac{1}{3}).$$

Hence

$$\delta_B = \frac{\ln^3 3}{2} + \frac{\pi^2\ln 3}{12} - \frac{3}{4}\zeta(3) - \mathrm{Li}_3\Big(-\frac{1}{3}\Big).$$

(3e) α_2: substituting $v = 1/w$ gives $A = \ln(1-v) - \ln v$, so

$$\alpha_2 = \int_{1/3}^1 \big[\ln(1-v) - \ln v\big]^2 \frac{dv}{v} = E_1 - 2\int_{1/3}^1 \frac{\ln v\ln(1-v)}{v}dv + \frac{\ln^3 3}{3},$$

$$E_1 := \int_{1/3}^1 \frac{\ln^2(1-v)}{v}\,dv.$$

By (A1) (endpoint limits at $v \to 1^-$ all vanish: $\ln^2(1-v)\ln v \to 0$, $\ln(1-v)\,\mathrm{Li}_2(1-v) \to 0$, $\mathrm{Li}_3(0) = 0$),

$$E_1 = \ln 3\, \ln^2 \frac{2}{3} - 2\ln\frac{2}{3}\; \mathrm{Li}_2\Big(\frac{2}{3}\Big) + 2\,\mathrm{Li}_3\Big(\frac{2}{3}\Big),$$

so, with (A2) as in (3c),

$$\alpha_2 = E_1 + \frac{\ln^3 3}{3} - 2\zeta(3) + 2\,\mathrm{Li}_3\Big(\frac{1}{3}\Big) + 2\ln 3\,\mathrm{Li}_2\Big(\frac{1}{3}\Big).$$

(3f) β_2: with $w = 3v$,

$$\beta_2 = \ln^3 3 + 2\ln 3\Big(\frac{\pi^2}{12} + \mathrm{Li}_2\Big(-\frac{1}{3}\Big)\Big) + \int_{1/3}^{1} \frac{\ln^2(1+v)}{v}\, dv,$$

$$\int_{1/3}^{1} \frac{\ln^2(1+v)}{v} dv \overset{\text{(A6)}}{=} F\Big(\frac{1}{2}\Big) - F\Big(\frac{1}{4}\Big).$$

Using (L7), a short computation gives

$$F\Big(\frac{1}{2}\Big) = -\ln^3 2 - 2\ln 2\,\mathrm{Li}_2\Big(\frac{1}{2}\Big) - 2\,\mathrm{Li}_3\Big(\frac{1}{2}\Big) + \frac{\ln^3 2}{3} = -\frac{7}{4}\zeta(3),$$

while $F(\frac{1}{4}) = \ln^2\frac{3}{4}\,\ln\frac{1}{4} + 2\ln\frac{3}{4}\,\mathrm{Li}_2(\frac{3}{4}) - 2\,\mathrm{Li}_3(\frac{3}{4}) - \frac{1}{3}\ln^3\frac{3}{4}$.

(3g) γ: substituting $v = 1/w$ ($A = \ln\frac{1-v}{v}$, $B = \ln\frac{1+3v}{v}$),

$$\gamma = \int_{1/3}^{1} \big[\ln(1-v) - \ln v\big]\big[\ln(1+3v) - \ln v\big]\frac{dv}{v} = H - \int_{1/3}^{1} \frac{\ln v\ln(1-v)}{v} dv - M + \frac{\ln^3 3}{3},$$

where $H := \int_{1/3}^{1} \frac{\ln(1-v)\ln(1+3v)}{v} dv$ and, by (A3) with $c = 3$,

$$M := \int_{1/3}^{1} \frac{\ln v\, \ln(1+3v)}{v}\, dv = \big[\,\mathrm{Li}_3(-3v) - \ln v\,\mathrm{Li}_2(-3v)\big]_{1/3}^{1} = \mathrm{Li}_3(-3) + \frac{3}{4}\zeta(3) + \frac{\pi^2\ln 3}{12}.$$

(3h) The mixed integral H by polarization and a Möbius involution. From $2PQ = P^2 + Q^2 - (P-Q)^2$,

$$H = \frac{1}{2}[E_1 + E_2 - R], \qquad E_2 := \int_{1/3}^{1} \frac{\ln^2(1+3v)}{v} dv, \qquad R := \int_{1/3}^{1} \ln^2\Big(\frac{1-v}{1+3v}\Big)\frac{dv}{v}.$$

For E_2, the substitution $z = 3v$ and (A6) give

$$E_2 = \int_{1}^{3} \frac{\ln^2(1+z)}{z}\, dz = F\Big(\frac{3}{4}\Big) - F\Big(\frac{1}{2}\Big) = F\Big(\frac{3}{4}\Big) + \frac{7}{4}\zeta(3),$$

$$F\Big(\frac{3}{4}\Big) = 4\ln^2 2\, \ln\frac{3}{4} - 4\ln 2\,\mathrm{Li}_2\Big(\frac{1}{4}\Big) - 2\,\mathrm{Li}_3\Big(\frac{1}{4}\Big) + \frac{8\ln^3 2}{3}.$$

For R, use the **involution** $v = \frac{1-y}{1+3y}$ (a self-inverse Möbius map exchanging $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$): then $\frac{1-v}{1+3v} = y$ (indeed $1 - v = \frac{4y}{1+3y}$, $1 + 3v = \frac{4}{1+3y}$), and $\frac{dv}{v} = d\ln v = (\frac{-1}{1-y} - \frac{3}{1+3y})dy$, so (orientation reverses)

$$R = \int_{0}^{1/3} \ln^2 y\left(\frac{1}{1-y} + \frac{3}{1+3y}\right) dy \overset{\text{(A4),(A5)}}{=} \ln^3 3 + 2\ln 3\,\mathrm{Li}_2\Big(\frac{1}{3}\Big) + 2\,\mathrm{Li}_3\Big(\frac{1}{3}\Big) + \frac{\pi^2\ln 3}{6} + \frac{3}{2}\zeta(3),$$

where the boundary values used are: at $y \to 0^+$ everything vanishes; at $y = \frac{1}{3}$, (A4) gives $\ln^3 3 - \ln 2\ln^2 3 + 2\ln 3\,\mathrm{Li}_2(\frac{1}{3}) + 2\,\mathrm{Li}_3(\frac{1}{3})$ and (A5) gives $\ln 2\ln^2 3 + \frac{\pi^2\ln 3}{6} + \frac{3}{2}\zeta(3)$ (by (V)).

(Each of (3a)–(3h) was verified independently by high-precision quadrature to 60 digits.)

4. Reduction table

The pieces above involve Li_2 at $\frac{2}{3}, \frac{1}{4}, \frac{3}{4}$ and Li_3 at $\frac{2}{3}, \frac{1}{4}, \frac{3}{4}, -3$. Reduce them to the basis

$$\left\{ \mathrm{Li}_2\left(\tfrac{1}{3}\right),\ \mathrm{Li}_2\left(-\tfrac{1}{3}\right),\ \mathrm{Li}_3\left(\tfrac{1}{3}\right),\ \mathrm{Li}_3\left(-\tfrac{1}{3}\right),\ \mathrm{Li}_3\left(-\tfrac{1}{2}\right)\right\}$$

by the toolkit identities (each line follows from the lemma cited, applied strictly inside its proven domain):

value	reduction	by
$\mathrm{Li}_2(\frac{2}{3})$	$\frac{\pi^2}{6} + \ln 3 \ln \frac{2}{3} - \mathrm{Li}_2(\frac{1}{3})$	(L1), $x = \frac{1}{3}$
$\mathrm{Li}_2(\frac{1}{4})$	$-\mathrm{Li}_2(-\frac{1}{3}) - \frac{1}{2}\ln^2 \frac{3}{4}$	(L2), $x = \frac{1}{4}$
$\mathrm{Li}_2(\frac{3}{4})$	$\frac{\pi^2}{6} + 2\ln 2 \ln \frac{3}{4} + \mathrm{Li}_2(-\frac{1}{3}) + \frac{1}{2}\ln^2 \frac{3}{4}$	(L1), $x = \frac{1}{4}$ + previous
$\mathrm{Li}_3(-2)$	$\mathrm{Li}_3(-\frac{1}{2}) - \frac{\pi^2 \ln 2}{6} - \frac{\ln^3 2}{6}$	(L4), $x = 2$
$\mathrm{Li}_3(-3)$	$\mathrm{Li}_3(-\frac{1}{3}) - \frac{\pi^2 \ln 3}{6} - \frac{\ln^3 3}{6}$	(L4), $x = 3$
$\mathrm{Li}_3(\frac{2}{3})$	$\zeta(3) + \frac{\ln^3 3}{3} - \frac{\pi^2 \ln 3}{6} - \frac{\ln 2 \ln^2 3}{2} - \mathrm{Li}_3(\frac{1}{3}) - \mathrm{Li}_3(-2)$	(L5), $x = \frac{1}{3}$ (where $1 - \frac{1}{x} = -2$)
$\mathrm{Li}_3(\frac{1}{4})$	$4\,\mathrm{Li}_3(\frac{1}{2}) + 4\,\mathrm{Li}_3(-\frac{1}{2}) = \frac{7}{2}\zeta(3) - \frac{\pi^2 \ln 2}{3} + \frac{2\ln^3 2}{3} + 4\,\mathrm{Li}_3(-\frac{1}{2})$	(L6), $z = \frac{1}{2}$; (L7)
$\mathrm{Li}_3(\frac{3}{4})$	$\zeta(3) + \frac{8\ln^3 2}{3} - \frac{\pi^2 \ln 2}{3} - 2\ln^2 2 \ln 3 - \mathrm{Li}_3(\frac{1}{4}) - \mathrm{Li}_3(-3)$	(L5), $x = \frac{1}{4}$ (where $1 - \frac{1}{x} = -3$)

5. Assembly: the native closed form

Substituting (3a)–(3h) and the reduction table into (2.1) and collecting terms is a finite computation in exact rational arithmetic (performed symbolically with a computer-algebra system, and independently confirmed numerically to 60 and then 170 digits). The result is the **native form**

$$\begin{aligned} I = &-\frac{23}{2}\zeta(3) + \frac{5\pi^2}{3}\ln 2 + \frac{10}{3}\ln^3 2 - \ln^3 3 - 6\ln^2 2 \ln 3 + 2\ln 2 \ln^2 3 \\ &- 4\,\mathrm{Li}_3\Big(\frac{1}{3}\Big) - 2\,\mathrm{Li}_3\Big(-\frac{1}{3}\Big) - 20\,\mathrm{Li}_3\Big(-\frac{1}{2}\Big) \\ &- 4\ln 3\ \mathrm{Li}_2\Big(\frac{1}{3}\Big) + 4\ln 2\ \mathrm{Li}_2\Big(-\frac{1}{3}\Big). \end{aligned} \tag{5.1}$$

This is already a complete closed form; the remaining sections only beautify it.

6. Two ladder identities

(R1) was proved in §0.2: $2\,\mathrm{Li}_2(\frac{1}{3}) - \mathrm{Li}_2(-\frac{1}{3}) = \frac{\pi^2}{6} - \frac{\ln^2 3}{2}$.

(R2) The weight-3 ladder.

$$2\,\mathrm{Li}_3\Big(\frac{1}{3}\Big) - \mathrm{Li}_3\Big(-\frac{1}{3}\Big) = \frac{13}{6}\zeta(3) + \frac{\ln^3 3}{6} - \frac{\pi^2}{6}\ln 3. \tag{R2}$$

Proof. Consider $\mathcal{J} = \displaystyle\int_0^{1/2} \frac{\ln^2(1-t^2)}{t}\,dt$ and evaluate it in two ways.

(i) The substitution $u = t^2$ ($\frac{dt}{t} = \frac{du}{2u}$) gives $\mathcal{J} = \frac{1}{2}P_2(\frac{1}{4})$, with P_2 from (A1).

(ii) Expanding $\ln^2(1-t^2) = \big[\ln(1-t) + \ln(1+t)\big]^2$ and using the polarization identity $2\ln(1-t)\ln(1+t) = \ln^2(1-t) + \ln^2(1+t) - \ln^2 \frac{1-t}{1+t}$:

$$\mathcal{J} = 2P_2(\tfrac{1}{2}) + 2G(\tfrac{1}{2}) - V, \qquad V := \int_0^{1/2} \ln^2\Big(\frac{1-t}{1+t}\Big)\frac{dt}{t},$$

with G from (A6′). For V, the involution $s=\frac{1-t}{1+t}$ (i.e. $t=\frac{1-s}{1+s}$, $\frac{dt}{t}=(\frac{-1}{1-s}-\frac{1}{1+s})ds$, mapping $t\in(0,\frac12)$ onto $s\in(\frac13,1)$ with reversed orientation) yields

$$V=\int_{1/3}^{1}\ln^2 s\left(\frac{1}{1-s}+\frac{1}{1+s}\right)ds$$
$$\overset{(A4),(A5)}{=}\Big[\frac72\zeta(3)\Big]-\Big[\ln^3 3-\ln 2\ln^2 3+2\ln 3\,\mathrm{Li}_2\left(\tfrac13\right)+2\,\mathrm{Li}_3\left(\tfrac13\right)\Big]$$
$$-\Big[\ln^2 3\ln\tfrac43-2\ln 3\,\mathrm{Li}_2\left(-\tfrac13\right)+2\,\mathrm{Li}_3\left(-\tfrac13\right)\Big].$$

Also, by (A1), (A6′), (L7):

$$P_2(\tfrac12)=\frac{\zeta(3)}{4}-\frac{\ln^3 2}{3},\qquad G(\tfrac12)=F(\tfrac13)+2\zeta(3),$$
$$F(\tfrac13)=\ln^2\tfrac23\ln\tfrac13+2\ln\tfrac23\,\mathrm{Li}_2\left(\tfrac23\right)-2\,\mathrm{Li}_3\left(\tfrac23\right)-\tfrac13\ln^3\tfrac23,$$
$$P_2(\tfrac14)=\ln^2\tfrac34\,\ln\tfrac14+2\ln\tfrac34\,\mathrm{Li}_2\left(\tfrac34\right)-2\,\mathrm{Li}_3\left(\tfrac34\right)+2\zeta(3).$$

Equating (i) and (ii), inserting the reduction table of §4, and collecting terms in exact arithmetic produces the relation

$$\begin{aligned}&(6\ln 3-4\ln 2)\,\mathrm{Li}_2\Big(\frac13\Big)+(2\ln 2-3\ln 3)\,\mathrm{Li}_2\Big(-\frac13\Big)+6\,\mathrm{Li}_3\Big(\frac13\Big)-3\,\mathrm{Li}_3\Big(-\frac13\Big)\\&=\frac{13}{2}\zeta(3)-\frac{\pi^2\ln 2}{3}+\ln 2\ln^2 3-\ln^3 3.\end{aligned}\tag{6.1}$$

Now subtract $(3\ln 3-2\ln 2)\times$(R1), i.e.

$$(6\ln 3-4\ln 2)\,\mathrm{Li}_2\Big(\frac13\Big)-(3\ln 3-2\ln 2)\,\mathrm{Li}_2\Big(-\frac13\Big)=\frac{\pi^2\ln 3}{2}-\frac{3\ln^3 3}{2}-\frac{\pi^2\ln 2}{3}+\ln 2\ln^2 3.$$

The Li_2 terms cancel completely, leaving $6\,\mathrm{Li}_3(\frac13)-3\,\mathrm{Li}_3(-\frac13)=\frac{13}{2}\zeta(3)+\frac{\ln^3 3}{2}-\frac{\pi^2\ln 3}{2}$, which is (R2). □

(Both (6.1) and (R2) were verified numerically to 170 digits. (R2) is the classical trilogarithm ladder equivalent to $\mathrm{Li}_3(\frac19)=4\,\mathrm{Li}_3(\frac13)+4\,\mathrm{Li}_3(-\frac13)$ *combined with* $3\,\mathrm{Li}_3(\frac13)-\frac14\,\mathrm{Li}_3(\frac19)=\frac{13}{6}\zeta(3)+\frac{\ln^3 3}{6}-\frac{\pi^2\ln 3}{6}$*.)*

7. Final form

Apply (R1) and (R2) to (5.1):

$$4\ln 2\ \mathrm{Li}_2\Big(-\frac13\Big)=8\ln 2\,\mathrm{Li}_2\Big(\frac13\Big)-\frac{2\pi^2\ln 2}{3}+2\ln 2\ln^2 3,$$
$$-2\,\mathrm{Li}_3\Big(-\frac13\Big)=-4\,\mathrm{Li}_3\Big(\frac13\Big)+\frac{13}{3}\zeta(3)-\frac{\pi^2\ln 3}{3}+\frac{\ln^3 3}{3}.$$

Substituting and collecting:

$$\boxed{\begin{aligned}I=\int_0^1\frac{\ln^2 x}{\sqrt{x^2-x+1}}\,dx&=4\ln\tfrac43\ \mathrm{Li}_2\Big(\tfrac13\Big)-8\,\mathrm{Li}_3\Big(\tfrac13\Big)-20\,\mathrm{Li}_3\Big(-\tfrac12\Big)-\frac{43}{6}\zeta(3)+\pi^2\ln 2\\&\quad-\frac{\pi^2}{3}\ln 3+\frac{10}{3}\ln^3 2-\frac23\ln^3 3-6\ln^2 2\ \ln 3+4\ln 2\ \ln^2 3\end{aligned}}$$

The negative-argument variant in the Result section follows the same way, trading $\mathrm{Li}_2(\frac13),\mathrm{Li}_3(\frac13)$ for $\mathrm{Li}_2(-\frac13),\mathrm{Li}_3(-\frac13)$ via (R1), (R2). Using (L6)+(L7) one may also replace $\mathrm{Li}_3(-\frac12)$ by $\frac14\,\mathrm{Li}_3(\frac14)-\mathrm{Li}_3(\frac12)$ if arguments in $(0,1)$ are preferred.

Numerical verification

All computations with mpmath at `mp.dps = 170`:

- Direct tanh–sinh quadrature of $\int_0^1 \frac{\ln^2 x}{\sqrt{x^2-x+1}}dx$ (split at $\frac{1}{4}, \frac{1}{2}$) and of the rationalized form (1.2) agree to 2.9×10^{-170}:

  $$I = 2.10029012483843065541358656514017065178479851127691422449913\ldots$$

  matching (and extending) the 54 digits quoted in the problem statement.

- The boxed closed form evaluates to the same number; $|I_{\text{quad}} - I_{\text{closed}}| \approx 2.1 \times 10^{-170}$, i.e. **agreement to 168+ significant digits** (the native form (5.1) and the negative-argument variant agree to the same precision).

- Every intermediate piece ($\alpha_1, \beta_1, \alpha_2, \beta_2, \delta_A, \delta_B, \gamma, E_1, E_2, R, H, M, \ldots$) was checked against direct quadrature to ~ 60 digits; the ladder identities (R1), (R2), Abel's identity (L8) (at interior test points) and the Landen equation (L5) were checked to 170 digits.

- The closed form was independently *discovered* by PSLQ (integer relation over the 12-element basis

  $$\{I, \zeta(3), \pi^2 \ln 2, \pi^2 \ln 3, \ln^3 2, \ln^3 3, \ln^2 2 \ln 3, \ln 2 \ln^2 3, \mathrm{Li}_3(\tfrac{1}{3}), \mathrm{Li}_3(-\tfrac{1}{2}), \ln 2\, \mathrm{Li}_2(\tfrac{1}{3}), \ln 3\, \mathrm{Li}_2(\tfrac{1}{3})\}$$

  at 160 digits, residual $\sim 10^{-159}$) and then *proved* as above — the derivation is logically independent of PSLQ.

Notes

- The derivation is complete and self-contained: the only inputs are the elementary antiderivatives (A1)–(A6), the classical functional equations (L1)–(L8) (each proved by differentiation plus one point evaluation, applied only on real intervals where all arguments stay in $(-\infty, 1]$ and the real-analytic branches are the ones occurring), the Euler substitution of §1, and the involution tricks of §3h/§6.

- Two collection steps (the assembly in §5 and the reduction (6.1)) involve straightforward but lengthy exact rational-linear algebra; they were executed with a computer-algebra system (sympy, exact arithmetic — no floating point) and double-checked numerically at 60–170 digits. Every input identity to those collections is proved in the text, so this is a matter of verified bookkeeping, not of mathematical gaps.

- The constants $\mathrm{Li}_3(\frac{1}{3})$ (equivalently $\mathrm{Li}_3(-\frac{1}{3})$ or $\mathrm{Li}_3(\frac{1}{9})$, modulo (R2)/(L6)) and $\mathrm{Li}_3(-\frac{1}{2})$ (equivalently $\mathrm{Li}_3(\frac{1}{4})$ or $\mathrm{Li}_3(\frac{3}{4})$) are not known to reduce to $\zeta(3)$ and elementary constants, and PSLQ searches at 160 digits found no such reduction; the boxed form is minimal in that sense.

- Bookkeeping identity worth recording: the mixed integral of §3h is $H = \int_{1/3}^1 \frac{\ln(1-v)\ln(1+3v)}{v}dv$, and the "two-ways" identity of §6 is what makes the base-$\frac{1}{3}$ trilogarithm ladder (R2) elementary — no appeal to Kummer's two-variable trilogarithm equation is needed.