Cleo Bench Problem 19

Infinite Series $\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n}$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$\Sigma_3 := \sum_{n=1}^{\infty} \frac{H_n}{n^3\, 2^n}, \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Result

$$\boxed{\sum_{n=1}^{\infty} \frac{H_n}{n^3\, 2^n} = \mathrm{Li}_4\!\left(\tfrac{1}{2}\right) + \frac{\pi^4}{720} - \frac{\ln 2}{8}\,\zeta(3) + \frac{\ln^4 2}{24}}$$

Numerically $\Sigma_3 = 0.55823730083320863825151737933247278596\ldots$ Equivalently, since $\zeta(4) = \pi^4/90$, the constant $\pi^4/720$ may be written $\zeta(4)/8$.

Derivation

Throughout, $\mathrm{Li}_s(x) = \sum_{n\ge1} x^n/n^s$ for $|x| \le 1$ (with $s \ge 2$ at $|x| = 1$), extended to real $x \le 0$ by the integral definitions

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\, dt, \qquad \mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\, dt,$$

which agree with the series on $(-1, 1)$ and are continuous on $[-1, 1)$; by Abel's theorem they agree with the (absolutely convergent) series also at $z = -1$. We write

$$\Sigma_k := \sum_{n\ge1} \frac{H_n}{n^k 2^n}.$$

We use the classical Euler values $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

Step 0: special polylogarithm values

(a) $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$**.** Splitting $\zeta(3)$ into even and odd n gives $\sum_{n\ge1} \frac{(-1)^{n-1}}{n^3} = (1 - 2^{1-3})\zeta(3) = \frac{3}{4}\zeta(3)$, so $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$.

*Problem originally posed on Mathematics Stack Exchange (question 909228, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

(b) Dilogarithm reflection. For $0 < x < 1$ let $f(x) = \mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln x \ln(1-x)$. Then

$$f'(x) = -\frac{\ln(1-x)}{x} + \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} - \frac{\ln x}{1-x} = 0,$$

and as $x \to 0^+$, $f(x) \to \zeta(2)$ (the product term is $O(x \ln x)$). Hence

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x) \qquad (0 < x < 1). \tag{0.1}$$

At $x = \frac{1}{2}$:

$$\mathrm{Li}_2\left(\tfrac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}. \tag{0.2}$$

(c) Landen's dilogarithm transformation. For $0 < x < 1$ both sides of

$$\mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\mathrm{Li}_2(x) - \frac{1}{2}\ln^2(1-x) \tag{0.3}$$

vanish at $x = 0$ and have equal derivatives: with $u = \frac{x}{x-1}$ one has $\frac{u'}{u} = \frac{1}{x} + \frac{1}{1-x}$ and $1 - u = \frac{1}{1-x}$, so the left derivative is $-\ln(1-u)\frac{u'}{u} = \ln(1-x)\left(\frac{1}{x} + \frac{1}{1-x}\right)$, which equals the right derivative $\frac{\ln(1-x)}{x} + \frac{\ln(1-x)}{1-x}$.

(d) Landen's trilogarithm identity and $\mathrm{Li}_3(1/2)$. For $0 < x < 1$ define

$$G(x) = \mathrm{Li}_3\left(\frac{x}{x-1}\right) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3(x),$$
$$R(x) = \zeta(3) + \frac{\pi^2}{6}\ln(1-x) - \frac{1}{2}\ln x \ln^2(1-x) + \frac{1}{6}\ln^3(1-x).$$

Using $\frac{d}{dx}\mathrm{Li}_3(u) = \frac{\mathrm{Li}_2(u)}{u}u'$ and (0.3),

$$G'(x) = \frac{\mathrm{Li}_2(x)}{x} - \frac{\mathrm{Li}_2(1-x)}{1-x} + \left[-\mathrm{Li}_2(x) - \tfrac{1}{2}\ln^2(1-x)\right]\left(\frac{1}{x} + \frac{1}{1-x}\right)$$
$$= -\frac{\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x)}{1-x} - \frac{\ln^2(1-x)}{2}\left(\frac{1}{x} + \frac{1}{1-x}\right),$$

and by the reflection (0.1),

$$G'(x) = -\frac{\zeta(2)}{1-x} + \frac{\ln x \ln(1-x)}{1-x} - \frac{\ln^2(1-x)}{2x} - \frac{\ln^2(1-x)}{2(1-x)}.$$

Differentiating R term by term gives exactly the same expression. Moreover $G(x) \to \zeta(3)$ and $R(x) \to \zeta(3)$ as $x \to 0^+$. Hence $G \equiv R$ on $(0,1)$. Putting $x = \frac{1}{2}$ (so $\frac{x}{x-1} = -1$, $\ln x = \ln(1-x) = -\ln 2$) and using (a):

$$2\,\mathrm{Li}_3\left(\tfrac{1}{2}\right) - \tfrac{3}{4}\zeta(3) = \zeta(3) - \frac{\pi^2 \ln 2}{6} + \frac{\ln^3 2}{2} - \frac{\ln^3 2}{6},$$

$$\mathrm{Li}_3\left(\tfrac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2 \ln 2}{12} + \frac{\ln^3 2}{6}. \tag{0.4}$$

Step 1: the generating-function ladder

For $|x| < 1$, the Cauchy product of the absolutely convergent series $\frac{1}{1-x} = \sum_{k\geq 0} x^k$ and $-\ln(1-x) = \sum_{m\geq 1} \frac{x^m}{m}$ gives

$$\sum_{n\geq 1} H_n x^n = \frac{-\ln(1-x)}{1-x}. \tag{1.1}$$

Power series may be integrated term by term inside the radius of convergence, so for $0 \leq x < 1$, dividing by t and integrating, with $\frac{1}{t(1-t)} = \frac{1}{t} + \frac{1}{1-t}$:

$$\sum_{n\geq 1} \frac{H_n}{n} x^n = \int_0^x \frac{-\ln(1-t)}{t}\,dt + \int_0^x \frac{-\ln(1-t)}{1-t}\,dt = \mathrm{Li}_2(x) + \frac{\ln^2(1-x)}{2}. \tag{1.2}$$

At $x = \frac{1}{2}$, by (0.2):

$$\Sigma_1 = \mathrm{Li}_2\left(\tfrac{1}{2}\right) + \frac{\ln^2 2}{2} = \frac{\pi^2}{12}. \tag{1.3}$$

Repeating (all integrands are $O(t)$ at 0, so the integrals converge):

$$\sum_{n\geq 1} \frac{H_n}{n^2} x^n = \mathrm{Li}_3(x) + \frac{1}{2}F(x), \qquad F(x) := \int_0^x \frac{\ln^2(1-t)}{t}\,dt, \tag{1.4}$$

$$\sum_{n\geq 1} \frac{H_n}{n^3} x^n = \mathrm{Li}_4(x) + \frac{1}{2}\int_0^x F(t)\,\frac{dt}{t}. \tag{1.5}$$

Since $F(t) = \frac{1}{2}t^2 + O(t^3)$ as $t \to 0^+$, we have $F(t)\ln t \to 0$, and integration by parts gives

$$\int_0^x F(t)\,\frac{dt}{t} = F(x)\ln x - \int_0^x \frac{\ln t\,\ln^2(1-t)}{t}\,dt.$$

Evaluating (1.5) at $x = \frac{1}{2}$:

$$\Sigma_3 = \mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{\ln 2}{2}A - \frac{1}{2}B, \qquad A := F\left(\tfrac{1}{2}\right), \qquad B := \int_0^{1/2} \frac{\ln t\,\ln^2(1-t)}{t}\,dt. \tag{1.6}$$

Step 2: the weight-3 integral A

Substituting $u = 1 - t$ and expanding $\frac{1}{1-u} = \sum_{n\geq 0} u^n$ on $[\frac{1}{2}, 1)$ (all terms nonnegative, so Tonelli's theorem justifies interchanging sum and integral):

$$A = \int_{1/2}^1 \frac{\ln^2 u}{1-u}\,du = \sum_{m\geq 1} \int_{1/2}^1 u^{m-1}\ln^2 u\,du.$$

From the antiderivative $\int u^{m-1}\ln^2 u\,du = u^m\left(\frac{\ln^2 u}{m} - \frac{2\ln u}{m^2} + \frac{2}{m^3}\right)$:

$$\int_0^1 u^{m-1}\ln^2 u\,du = \frac{2}{m^3}, \qquad \int_0^{1/2} u^{m-1}\ln^2 u\,du = \frac{\ln^2 2}{m\,2^m} + \frac{2\ln 2}{m^2\,2^m} + \frac{2}{m^3\,2^m}. \tag{2.1}$$

Hence

$$A = 2\zeta(3) - \ln^3 2 - 2\ln 2\ \mathrm{Li}_2\left(\tfrac{1}{2}\right) - 2\,\mathrm{Li}_3\left(\tfrac{1}{2}\right) \overset{(0.2),(0.4)}{=} \frac{\zeta(3)}{4} - \frac{\ln^3 2}{3}. \tag{2.2}$$

As a byproduct, (1.4) at $x = \frac{1}{2}$ gives

$$\Sigma_2 = \mathrm{Li}_3\left(\tfrac{1}{2}\right) + \frac{A}{2} = \zeta(3) - \frac{\pi^2\ln 2}{12}. \tag{2.3}$$

Step 3: Euler's theorem $\sum_{n\ge1}\frac{H_n}{n^3}=\frac54\zeta(4)$

Applying $\frac{1}{mn}=\frac{1}{m+n}\left(\frac1m+\frac1n\right)$ twice yields the algebraic identity, valid for all $m,n\ge1$:

$$\frac{1}{m^2n^2}=\frac{1}{(m+n)^2}\left(\frac{1}{m^2}+\frac{1}{n^2}\right)+\frac{2}{(m+n)^3}\left(\frac1m+\frac1n\right).$$

Summing over all $m,n\ge1$ (all terms positive, so Tonelli permits any order of summation):

- the left side is $\zeta(2)^2$;
- the first right-hand piece equals, by symmetry and $N:=m+n$, $2\sum_{m\ge1}\sum_{N>m}\frac{1}{m^2N^2}=2\sum_{m<N}\frac{1}{m^2N^2}=\zeta(2)^2-\zeta(4)$, since $\zeta(2)^2=2\sum_{m<N}\frac{1}{m^2N^2}+\zeta(4)$;
- the second piece equals $4\sum_{m\ge1}\sum_{N>m}\frac{1}{m\,N^3}=4\sum_{N\ge2}\frac{H_{N-1}}{N^3}=4\Big(\sum_{N\ge1}\frac{H_N}{N^3}-\zeta(4)\Big)$.

Therefore $\zeta(2)^2=\zeta(2)^2-\zeta(4)+4\sum_N\frac{H_N}{N^3}-4\zeta(4)$, i.e.

$$\sum_{n\ge1}\frac{H_n}{n^3}=\frac54\,\zeta(4)=\frac{\pi^4}{72}. \tag{3.1}$$

Step 4: the full-range weight-4 integral

Let $I_1:=\int_0^1\frac{\ln t\,\ln^2(1-t)}{t}\,dt$ (convergent: the integrand is $O(t\ln t)$ at 0 and $O(\ln^2(1-t))$ at 1). Substituting $t\mapsto1-t$ and expanding $-\frac{\ln(1-t)}{1-t}=\sum_{n\ge1}H_nt^n$ by (1.1) (integrand of one fixed sign on $(0,1)$: Tonelli):

$$I_1=\int_0^1\frac{\ln^2t\,\ln(1-t)}{1-t}\,dt=-\sum_{n\ge1}H_n\int_0^1t^n\ln^2t\,dt=-2\sum_{n\ge1}\frac{H_n}{(n+1)^3}.$$

Reindexing with $H_n=H_{n+1}-\frac{1}{n+1}$, $\sum_{n\ge1}\frac{H_n}{(n+1)^3}=\sum_{m\ge1}\frac{H_m}{m^3}-\zeta(4)$, so by (3.1)

$$I_1=-2\left(\frac54\zeta(4)-\zeta(4)\right)=-\frac{\zeta(4)}{2}=-\frac{\pi^4}{180}. \tag{4.1}$$

Step 5: reflecting B back to the sums Σ_k

Split $I_1=B+\int_{1/2}^1\frac{\ln t\ln^2(1-t)}{t}dt$ and substitute $t=1-u$ in the second piece:

$$B=I_1-C,\qquad C:=\int_0^{1/2}\frac{\ln^2u\,\ln(1-u)}{1-u}\,du. \tag{5.1}$$

Expanding $-\frac{\ln(1-u)}{1-u}=\sum_{n\ge1}H_nu^n$ on $(0,\frac12)$ (again one fixed sign: Tonelli) and using (2.1) with $m=n+1$:

$$-C=\sum_{n\ge1}H_n\left[\frac{\ln^22}{(n+1)2^{n+1}}+\frac{2\ln2}{(n+1)^22^{n+1}}+\frac{2}{(n+1)^32^{n+1}}\right].$$

For $k=1,2,3$ the reindex $m=n+1$, $H_n=H_m-\frac1m$ gives

$$\sum_{n\ge1}\frac{H_n}{(n+1)^k2^{n+1}}=\sum_{m\ge1}\frac{H_m}{m^k2^m}-\sum_{m\ge1}\frac{1}{m^{k+1}2^m}=\Sigma_k-\mathrm{Li}_{k+1}\!\left(\tfrac12\right),$$

(the $m=1$ terms of the two shifted series cancel because $H_0=0$). Hence

$$C=-\ln^22\left[\Sigma_1-\mathrm{Li}_2\!\left(\tfrac12\right)\right]-2\ln2\left[\Sigma_2-\mathrm{Li}_3\!\left(\tfrac12\right)\right]-2\left[\Sigma_3-\mathrm{Li}_4\!\left(\tfrac12\right)\right]. \tag{5.2}$$

Step 6: assembly

Insert (5.1)–(5.2) into (1.6):

$$\begin{aligned}\Sigma_3 &= \mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{\ln 2}{2}A - \frac{I_1}{2} + \frac{C}{2}\\ &= \mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{\ln 2}{2}A - \frac{I_1}{2} - \frac{\ln^2 2}{2}\Big[\Sigma_1 - \mathrm{Li}_2\left(\tfrac{1}{2}\right)\Big]\\ &\quad - \ln 2\Big[\Sigma_2 - \mathrm{Li}_3\left(\tfrac{1}{2}\right)\Big] - \Sigma_3 + \mathrm{Li}_4\left(\tfrac{1}{2}\right).\end{aligned}$$

The unknown Σ_3 occurs on the right with coefficient -1 (this is where the reflection $t \mapsto 1-t$, anchored by the independently evaluated I_1, breaks the circularity), so

$$2\Sigma_3 = 2\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{\ln 2}{2}A - \frac{I_1}{2} - \frac{\ln^2 2}{2}\Sigma_1 + \frac{\ln^2 2}{2}\,\mathrm{Li}_2\left(\tfrac{1}{2}\right) - \ln 2\,\Sigma_2 + \ln 2\;\mathrm{Li}_3\left(\tfrac{1}{2}\right).$$

Now substitute the values (2.2), (4.1), (1.3), (0.2), (2.3), (0.4):

$$\begin{aligned} -\frac{\ln 2}{2}A &= -\frac{\zeta(3)\ln 2}{8} + \frac{\ln^4 2}{6}, & -\frac{I_1}{2} &= \frac{\pi^4}{360},\\ -\frac{\ln^2 2}{2}\Sigma_1 &= -\frac{\pi^2\ln^2 2}{24}, & \frac{\ln^2 2}{2}\,\mathrm{Li}_2\left(\tfrac{1}{2}\right) &= \frac{\pi^2\ln^2 2}{24} - \frac{\ln^4 2}{4},\\ -\ln 2\,\Sigma_2 &= -\zeta(3)\ln 2 + \frac{\pi^2\ln^2 2}{12}, & \ln 2\;\mathrm{Li}_3\left(\tfrac{1}{2}\right) &= \frac{7\zeta(3)\ln 2}{8} - \frac{\pi^2\ln^2 2}{12} + \frac{\ln^4 2}{6}.\end{aligned}$$

The $\pi^2\ln^2 2$ terms cancel; the $\zeta(3)\ln 2$ coefficients sum to $-\frac{1}{8}-1+\frac{7}{8} = -\frac{1}{4}$; the $\ln^4 2$ coefficients to $\frac{1}{6} - \frac{1}{4} + \frac{1}{6} = \frac{1}{12}$. Hence

$$2\Sigma_3 = 2\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \frac{\pi^4}{360} - \frac{\zeta(3)\ln 2}{4} + \frac{\ln^4 2}{12},$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3\,2^n} = \mathrm{Li}_4\left(\tfrac{1}{2}\right) + \frac{\pi^4}{720} - \frac{\ln 2}{8}\,\zeta(3) + \frac{\ln^4 2}{24}. \qquad \blacksquare$$

(The final algebraic assembly was independently re-done symbolically in sympy and agrees.)

Numerical verification

All computations with mpmath.

- **Direct value** (brute-force partial sum of 300 terms at 80-digit working precision; the tail is $< \sum_{n>300} 2^{-n} < 10^{-90}$):

  $$\Sigma_3 = 0.55823730083320863825151737933247278596235920552729612849032191624\ldots$$

- **Closed form** $\mathrm{Li}_4(\frac{1}{2}) + \frac{\pi^4}{720} - \frac{\ln 2}{8}\zeta(3) + \frac{\ln^4 2}{24}$:

  $$0.55823730083320863825151737933247278596235920552729612849032191624\ldots$$

- **Agreement:** absolute difference 4.2×10^{-81}, i.e. **80 significant digits** agree.

Every intermediate identity was verified independently at 60-digit precision (each residual $< 10^{-60}$ unless noted): $\Sigma_1 = \pi^2/12$; $\Sigma_2 = \zeta(3) - \frac{\pi^2 \ln 2}{12}$; $\mathrm{Li}_2(\frac{1}{2})$ and $\mathrm{Li}_3(\frac{1}{2})$ closed forms (0.2), (0.4); $A = \frac{\zeta(3)}{4} - \frac{\ln^3 2}{3}$ (via `mp.quad`); $I_1 = -\pi^4/180$ (via `mp.quad`); the splitting $B + C = I_1$; the series representation (5.2) of C; the integration-by-parts identity (1.6); and Euler's sum (3.1) (partial sum to $N = 2 \cdot 10^5$ plus Euler–Maclaurin tail, residual 8×10^{-16}, consistent with the tail order used — and (3.1) is proved rigorously in Step 3 in any case).

Scripts: `verify.py`, `verify2.py` in the working directory `/tmp/claude-1000/-home-riv-Code-cleo-bench/817bd907-85a0-4bc2-a728-b3544bb304b6/scratchpad/cleo/work/q909228-infinite-series-sum-n-1-infty-frac-h-n-n-32-n/`.

Notes

- The derivation is self-contained modulo two classical facts used without proof: $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$ (Euler). Everything else — the reflection and Landen identities for Li_2 and Li_3, the values $\mathrm{Li}_2(\frac{1}{2})$, $\mathrm{Li}_3(\frac{1}{2})$, $\mathrm{Li}_3(-1)$, Euler's linear sum $\sum H_n/n^3 = \frac{5}{4}\zeta(4)$, and all integral evaluations — is proved above.
- All interchanges of summation and integration are justified by Tonelli's theorem (fixed-sign integrands) or by uniform convergence of power series on compact subsets of the disc of convergence; each such point is flagged where it occurs.
- $\mathrm{Li}_4(\frac{1}{2})$ is not expressible in terms of π, $\ln 2$, $\zeta(3)$ alone (it is standardly taken as an independent constant at weight 4), so the answer above is the expected minimal closed form.
- No caveats: the derivation is complete and the numerics agree to 80 digits.