

Cleo Bench Problem 10

Closed form for $\int_{-\infty}^0 \text{Ei}^3 x \, dx$

Derivation by Claude (Fable 5), closed-book*

July 2026

Problem

Let $\text{Ei } x$ denote the exponential integral,

$$\text{Ei } x = - \int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt,$$

(for $x < 0$ the integral is a proper convergent one; that is the only range we need). Given the elementary facts

$$\int \text{Ei } x \, dx = x \text{Ei } x - e^x, \quad \int \text{Ei}^2 x \, dx = x \text{Ei}^2 x - 2e^x \text{Ei } x + 2 \text{Ei}(2x),$$

$$\int_{-\infty}^0 \text{Ei } x \, dx = -1, \quad \int_{-\infty}^0 \text{Ei}^2 x \, dx = \ln 4,$$

the question asks whether these results generalize to higher powers, in particular whether there are closed forms for

$$(6) \quad \int \text{Ei}^3 x \, dx \quad \text{or} \quad (7) \quad \int_{-\infty}^0 \text{Ei}^3 x \, dx.$$

The requested quantity (the title of the question) is (7).

Result

$$\boxed{\int_{-\infty}^0 \text{Ei}^3 x \, dx = -6 \ln^2 2 - 3 \text{Li}_2\left(\frac{1}{4}\right)}$$

with the equivalent rewritings (all proved below)

$$\begin{aligned} \int_{-\infty}^0 \text{Ei}^3 x \, dx &= -\frac{\pi^2}{2} - 3 \ln^2 2 - 6 \text{Li}_2\left(-\frac{1}{2}\right) \\ &= 6 \text{Li}_2\left(\frac{1}{3}\right) + 3 \ln^2 3 - 6 \ln 2 \ln 3 - \frac{\pi^2}{2} \\ &= \text{Li}_2\left(\frac{1}{9}\right) + 2 \ln^2 3 - 6 \ln 2 \ln 3 - \frac{\pi^2}{6}, \end{aligned}$$

*Problem originally posed on Mathematics Stack Exchange ([question 714628](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

where $\text{Li}_2(z) = \sum_{n \geq 1} z^n/n^2$. Numerically

$$\int_{-\infty}^0 \text{Ei}^3 x \, dx = -3.6856760007574063687601666434234245578428889103088373027967888611 \dots$$

For the indefinite integral (6), integration by parts gives

$$\int \text{Ei}^3 x \, dx = x \text{Ei}^3 x - 3e^x \text{Ei}^2 x + 6 \int \frac{e^{2x} \text{Ei} x}{x} \, dx,$$

and the remaining primitive $\int e^{2x} \text{Ei}(x) \, dx/x$ does not appear to be expressible through elementary functions and Ei (no proof of impossibility is claimed here); see the Notes. The definite integral (7), however, has the exact closed form above.

Derivation

0. Notation and reduction to E_1

For $s > 0$ put

$$E_1(s) = \int_s^\infty \frac{e^{-t}}{t} \, dt.$$

For $x < 0$ the definition of Ei gives directly $\text{Ei}(x) = -E_1(-x)$. Substituting $x = -s$,

$$\Omega := \int_{-\infty}^0 \text{Ei}^3 x \, dx = - \int_0^\infty E_1(s)^3 \, ds. \quad (0.1)$$

Two elementary bounds for E_1 will be used repeatedly:

- For $0 < s \leq 1$:

$$E_1(s) = \int_s^1 \frac{e^{-t}}{t} \, dt + E_1(1) \leq \int_s^1 \frac{dt}{t} + E_1(1) \leq \ln \frac{1}{s} + 1. \quad (0.2)$$

- For $s > 0$:

$$E_1(s) \leq \frac{1}{s} \int_s^\infty e^{-t} \, dt = \frac{e^{-s}}{s}. \quad (0.3)$$

By (0.2) the integrand in (0.1) is $O(\ln^3(1/s))$ near 0 and by (0.3) it is $O(e^{-3s}/s^3)$ at infinity, so the integral converges absolutely.

1. Integration by parts

E_1 is C^1 on $(0, \infty)$ with $E_1'(s) = -e^{-s}/s$ (fundamental theorem of calculus). Hence on any $[\varepsilon, N] \subset (0, \infty)$,

$$\frac{d}{ds} (s E_1(s)^3) = E_1(s)^3 + 3s E_1(s)^2 \left(-\frac{e^{-s}}{s} \right) = E_1(s)^3 - 3e^{-s} E_1(s)^2,$$

so

$$\int_\varepsilon^N E_1^3 \, ds = \left[s E_1(s)^3 \right]_\varepsilon^N + 3 \int_\varepsilon^N e^{-s} E_1(s)^2 \, ds.$$

By (0.3), $N E_1(N)^3 \leq e^{-3N}/N^2 \rightarrow 0$ as $N \rightarrow \infty$; by (0.2), $\varepsilon E_1(\varepsilon)^3 \leq \varepsilon (\ln \frac{1}{\varepsilon} + 1)^3 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Both integrals converge absolutely (same bounds), so

$$\int_0^\infty E_1(s)^3 \, ds = 3 \int_0^\infty e^{-s} E_1(s)^2 \, ds =: 3K. \quad (1.1)$$

2. Kernel representation and Tonelli

Substituting $t = su$ ($u > 1$) in the definition of E_1 ,

$$E_1(s) = \int_1^\infty \frac{e^{-su}}{u} du \quad (s > 0). \quad (2.1)$$

Therefore

$$e^{-s} E_1(s)^2 = \int_1^\infty \int_1^\infty \frac{e^{-s(1+u+v)}}{uv} du dv,$$

and the map $(s, u, v) \mapsto e^{-s(1+u+v)}/(uv)$ is nonnegative and measurable on $(0, \infty) \times (1, \infty)^2$. By Tonelli's theorem the triple integral may be computed iteratedly in any order; integrating over s first ($\int_0^\infty e^{-cs} ds = 1/c$),

$$K = \int_1^\infty \int_1^\infty \frac{du dv}{uv(1+u+v)}. \quad (2.2)$$

(The iterated evaluation below produces a finite value, which retroactively confirms integrability; Tonelli needs no integrability hypothesis for nonnegative integrands.)

3. The inner integral

For $c > 0$, partial fractions give $\frac{1}{v(v+c)} = \frac{1}{c} \left(\frac{1}{v} - \frac{1}{v+c} \right)$, whence

$$\int_1^\infty \frac{dv}{v(v+c)} = \frac{1}{c} \left[\ln \frac{v}{v+c} \right]_1^\infty = \frac{\ln(1+c)}{c}.$$

With $c = 1 + u$ this turns (2.2) into a single integral:

$$K = \int_1^\infty \frac{\ln(2+u)}{u(1+u)} du. \quad (3.1)$$

(Convergent: integrand $\sim \ln u/u^2$ at infinity.)

4. Rationalizing substitution

Let $t = \frac{1}{1+u}$, a smooth decreasing bijection $(1, \infty) \rightarrow (0, \frac{1}{2})$, i.e. $u = \frac{1-t}{t}$. Then

$$du = -\frac{dt}{t^2}, \quad 2+u = \frac{1+t}{t}, \quad u(1+u) = \frac{1-t}{t} \cdot \frac{1}{t} = \frac{1-t}{t^2},$$

so

$$K = \int_0^{1/2} \frac{\ln(1+t) - \ln t}{1-t} dt. \quad (4.1)$$

Both pieces below converge separately ($1-t \geq \frac{1}{2}$ on $[0, \frac{1}{2}]$, and $-\ln t$ is integrable at 0).

5. The two elementary pieces

Throughout, $\text{Li}_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-w)}{w} dw$ for $z \leq 1$ (the integral representation follows by termwise integration of the series of $-\ln(1-w)/w$, justified on $[0, z]$, $0 \leq z < 1$, by uniform convergence, and extends to $z = 1$ by monotone convergence). In particular

$$\frac{d}{dz} \text{Li}_2(z) = -\frac{\ln(1-z)}{z} \quad (z < 1). \quad (5.1)$$

(a) On $[0, \frac{1}{2}]$ expand $\frac{1}{1-t} = \sum_{n \geq 0} t^n$; every term $t^n (-\ln t) \geq 0$, so by monotone convergence (Tonelli for series),

$$\int_0^{1/2} \frac{-\ln t}{1-t} dt = \sum_{n \geq 0} \int_0^{1/2} t^n (-\ln t) dt = \sum_{n \geq 0} \left(\frac{\ln 2}{(n+1)2^{n+1}} + \frac{1}{(n+1)2^{2n+1}} \right) = \ln^2 2 + \text{Li}_2\left(\frac{1}{2}\right),$$

using $\int_0^a t^n (-\ln t) dt = \frac{a^{n+1}}{n+1} \ln \frac{1}{a} + \frac{a^{n+1}}{(n+1)^2}$ and $\sum_{m \geq 1} \frac{(1/2)^m}{m} = \ln 2$.

(b) Substitute $v = 1 - t$ (so $1 + t = 2 - v$):

$$\int_0^{1/2} \frac{\ln(1+t)}{1-t} dt = \int_{1/2}^1 \frac{\ln(2-v)}{v} dv = \int_{1/2}^1 \frac{\ln 2}{v} dv + \int_{1/2}^1 \frac{\ln(1-\frac{v}{2})}{v} dv.$$

The first integral is $\ln^2 2$. For the second, by (5.1) $\frac{d}{dv} \text{Li}_2(v/2) = -\ln(1-\frac{v}{2})/v$ (valid since $v/2 \in [\frac{1}{4}, \frac{1}{2}]$), so it equals $[-\text{Li}_2(v/2)]_{1/2}^1 = \text{Li}_2(\frac{1}{4}) - \text{Li}_2(\frac{1}{2})$. Hence

$$\int_0^{1/2} \frac{\ln(1+t)}{1-t} dt = \ln^2 2 - \text{Li}_2\left(\frac{1}{2}\right) + \text{Li}_2\left(\frac{1}{4}\right).$$

Adding (a) and (b), the $\text{Li}_2(\frac{1}{2})$ terms cancel:

$$K = 2 \ln^2 2 + \text{Li}_2\left(\frac{1}{4}\right). \quad (5.2)$$

6. Conclusion

Combining (0.1), (1.1), (5.2):

$$\begin{aligned} \int_{-\infty}^0 \text{Ei}^3 x dx &= -3K = -6 \ln^2 2 - 3 \text{Li}_2\left(\frac{1}{4}\right) \\ &= -3.685676000757406368760166643423424557842888910 \dots \end{aligned}$$

Note that the main result uses **no** dilogarithm identity at all — only the definition of Li_2 .

7. Equivalent forms

The following classical identities are proved from scratch; each “differentiation proof” uses (5.1) and checks the constant at one point.

(i) **Duplication.** For $|z| < 1$, from the absolutely convergent series,

$$\text{Li}_2(z) + \text{Li}_2(-z) = 2 \sum_{n \text{ even}} \frac{z^n}{n^2} = \frac{1}{2} \text{Li}_2(z^2).$$

At $z = \frac{1}{2}$: $\text{Li}_2(\frac{1}{4}) = 2 \text{Li}_2(\frac{1}{2}) + 2 \text{Li}_2(-\frac{1}{2})$.

(ii) **Euler’s value.** For $0 < x < 1$ let $R(x) = \text{Li}_2(x) + \text{Li}_2(1-x) + \ln x \ln(1-x)$. By (5.1),

$$R'(x) = -\frac{\ln(1-x)}{x} + \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} - \frac{\ln x}{1-x} = 0,$$

so R is constant; letting $x \rightarrow 0^+$ gives $R \equiv \text{Li}_2(1) = \zeta(2) = \pi^2/6$ (Basel’s classical evaluation).

At $x = \frac{1}{2}$: $\text{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$.

From (i)+(ii): $3 \text{Li}_2(\frac{1}{4}) = \frac{\pi^2}{2} - 3 \ln^2 2 + 6 \text{Li}_2(-\frac{1}{2})$, hence

$$\Omega = -6 \ln^2 2 - 3 \text{Li}_2\left(\frac{1}{4}\right) = -\frac{\pi^2}{2} - 3 \ln^2 2 - 6 \text{Li}_2\left(-\frac{1}{2}\right).$$

(iii) Landen's transformation. For $0 \leq x < 1$ let $L(x) = \text{Li}_2\left(\frac{-x}{1-x}\right) + \text{Li}_2(x) + \frac{1}{2} \ln^2(1-x)$. With $w = \frac{-x}{1-x}$ one has $1-w = \frac{1}{1-x}$ and $w' = -\frac{1}{(1-x)^2}$, so by (5.1)

$$\frac{d}{dx} \text{Li}_2(w) = -\frac{\ln(1-w)}{w} w' = -\frac{(1-x) \ln(1-x)}{x} \cdot \left(-\frac{1}{(1-x)^2}\right) \cdot (-1)^0 = \frac{\ln(1-x)}{x(1-x)},$$

hence

$$L'(x) = \frac{\ln(1-x)}{x(1-x)} - \frac{\ln(1-x)}{x} - \frac{\ln(1-x)}{1-x} = \ln(1-x) \left[\frac{1}{x(1-x)} - \frac{1}{x} - \frac{1}{1-x} \right] = 0,$$

and $L(0) = 0$, i.e. $\text{Li}_2\left(\frac{-x}{1-x}\right) = -\text{Li}_2(x) - \frac{1}{2} \ln^2(1-x)$. At $x = \frac{1}{3}$ (so $\frac{-x}{1-x} = -\frac{1}{2}$):

$$\text{Li}_2\left(-\frac{1}{2}\right) = -\text{Li}_2\left(\frac{1}{3}\right) - \frac{1}{2} \ln^2 \frac{3}{2}.$$

Substituting into the previous display, and using $3 \ln^2 \frac{3}{2} - 3 \ln^2 2 = 3 \ln^2 3 - 6 \ln 2 \ln 3$,

$$\Omega = 6 \text{Li}_2\left(\frac{1}{3}\right) + 3 \ln^2 3 - 6 \ln 2 \ln 3 - \frac{\pi^2}{2}.$$

(iv) A squaring identity and Landen's $\frac{1}{3}-\frac{1}{9}$ ladder. For $0 \leq x < \frac{1}{2}$ let

$$S(x) = 2 \text{Li}_2\left(\frac{x}{1-x}\right) - 2 \text{Li}_2(x) - \text{Li}_2\left(\frac{x^2}{(1-x)^2}\right) - \ln^2(1-x).$$

With $w = \frac{x}{1-x}$: $1-w = \frac{1-2x}{1-x}$, $w' = \frac{1}{(1-x)^2}$, so $\frac{d}{dx} \text{Li}_2(w) = -\frac{1}{x(1-x)} \ln \frac{1-2x}{1-x}$; with $W = w^2$: $1-W = \frac{1-2x}{(1-x)^2}$, $W' = \frac{2x}{(1-x)^3}$, so $\frac{d}{dx} \text{Li}_2(W) = -\frac{2}{x(1-x)} \ln \frac{1-2x}{(1-x)^2}$. Hence

$$\begin{aligned} S'(x) &= \frac{2}{x(1-x)} \left[\ln \frac{1-2x}{(1-x)^2} - \ln \frac{1-2x}{1-x} \right] + \frac{2 \ln(1-x)}{x} + \frac{2 \ln(1-x)}{1-x} \\ &= -\frac{2 \ln(1-x)}{x(1-x)} + \frac{2 \ln(1-x)}{x(1-x)} = 0, \end{aligned}$$

and $S(0) = 0$. At $x = \frac{1}{4}$:

$$2 \text{Li}_2\left(\frac{1}{3}\right) - 2 \text{Li}_2\left(\frac{1}{4}\right) - \text{Li}_2\left(\frac{1}{9}\right) = \ln^2 \frac{3}{4}.$$

Combining this with (i)–(iii) (eliminate $\text{Li}_2\left(\frac{1}{4}\right)$ via $\text{Li}_2\left(\frac{1}{4}\right) = \frac{\pi^2}{6} - \ln^2 2 - 2 \text{Li}_2\left(\frac{1}{3}\right) - \ln^2 \frac{3}{2}$, which follows from (i),(ii),(iii)) and simplifying the logarithms with $2 \ln^2 2 + 2 \ln^2 \frac{3}{2} - \ln^2 \frac{3}{4} = \ln^2 3$ yields Landen's ladder

$$6 \text{Li}_2\left(\frac{1}{3}\right) - \text{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{3} - \ln^2 3,$$

and therefore

$$\Omega = \text{Li}_2\left(\frac{1}{9}\right) + 2 \ln^2 3 - 6 \ln 2 \ln 3 - \frac{\pi^2}{6}.$$

$\text{Li}_2\left(\frac{1}{4}\right)$ (equivalently $\text{Li}_2\left(-\frac{1}{2}\right)$, $\text{Li}_2\left(\frac{1}{3}\right)$, $\text{Li}_2\left(\frac{1}{9}\right)$) — all four are linearly related over $\mathbb{Q}[\pi^2, \ln 2, \ln 3]$ by (i)–(iv)) is not known to reduce to elementary constants, so a genuine dilogarithm value remains in the answer.

8. Consistency of the method with the known cases

The same three-step scheme reproduces the values quoted in the problem:

- $n = 1$: $\int_0^\infty E_1 ds = [sE_1]_0^\infty + \int_0^\infty e^{-s} ds = 1$, so $\int_{-\infty}^0 \text{Ei } x dx = -1$ — formula (4). ✓
- $n = 2$: $\int_0^\infty E_1^2 ds = 2 \int_0^\infty e^{-s} E_1 ds = 2 \int_1^\infty \frac{du}{u(1+u)} = 2 \ln 2$, and since $(-E_1)^2 = E_1^2$, $\int_{-\infty}^0 \text{Ei}^2 x dx = + \int_0^\infty E_1^2 ds = \ln 4$ — formula (5). ✓

9. Remark on the indefinite integral (6)

Differentiating ($\text{Ei}'(x) = e^x/x$ for $x \neq 0$),

$$\frac{d}{dx} \left[x \text{Ei}^3 x - 3e^x \text{Ei}^2 x \right] = \text{Ei}^3 x - \frac{6 e^{2x} \text{Ei} x}{x},$$

so

$$\int \text{Ei}^3 x dx = x \text{Ei}^3 x - 3e^x \text{Ei}^2 x + 6 \int \frac{e^{2x} \text{Ei} x}{x} dx.$$

For the square, the analogous step produced $\int \frac{e^{2x}}{x} dx = \text{Ei}(2x)$, which closes in terms of Ei; for the cube one needs $\int e^{2x} \text{Ei}(x) dx/x$, a genuinely two-scale primitive for which no expression in elementary functions and Ei (of any arguments) is known. We do not prove non-existence; but this is why the pattern of (2)–(3) stops, while the definite integral (7) still evaluates in closed form. (Note also that splitting the definite integral along this identity is illegitimate: $e^x \text{Ei}^2 x \sim \ln^2 |x| \rightarrow +\infty$ as $x \rightarrow 0^-$ and $e^{2x} \text{Ei}(x)/x \sim \ln |x|/x$ is non-integrable at 0^- ; the divergences cancel only in the combination above. Our route via E_1 in Step 1 keeps all boundary terms finite.)

Numerical verification

Script: /tmp/claude-1000/-home-riv-Code-cleo-bench/817bd907-85a0-4bc2-a728-b3544bb304b6/scratchpad/cleo/work/q714628-closed-form-for-int-infty-0-operator-name-ei-3x-d/verify.py, run with mpmath at mp.dps = 80 (and re-checked at mp.dps = 90).

Two independent direct quadratures (tanh–sinh with split points; the integrand has only an integrable \ln^3 endpoint singularity at 0 and exponential decay at infinity):

- $\int_{-\infty}^0 \text{Ei}(x)^3 dx$ via `mp.ei`:
 $-3.6856760007574063687601666434234245578428889103088373027967888611$
 (est. quadrature error $\sim 10^{-84}$);
- $-\int_0^{\infty} E_1(s)^3 ds$ via `mp.e1`: identical to all 80 digits.

Closed form `-6*mp.log(2)**2 - 3*mp.polylog(2, mpf(1)/4)`:

$$-3.6856760007574063687601666434234245578428889103088373027967888611$$

Agreement: all 80 working digits (difference 0 at 80 dps; at 90 dps the difference is $6.4 \cdot 10^{-90}$), far beyond the required 25. The three alternative closed forms agree with the first to all 80 digits as well.

Every intermediate step of the derivation was also checked numerically to ~ 80 digits:

- the integration-by-parts identity $\int_0^{\infty} E_1^3 = 3 \int_0^{\infty} e^{-s} E_1^2$ (residual $\sim 10^{-81}$),
- $K = \int_1^{\infty} \frac{\ln(2+u)}{u(1+u)} du = \int_0^{1/2} \frac{\ln(1+t)-\ln t}{1-t} dt = 2 \ln^2 2 + \text{Li}_2(\frac{1}{4})$ (residuals 0 at 80 dps),
- the pieces 5(a), 5(b) separately (residuals 0),
- the dilogarithm identities (i)–(iv) and the $\frac{1}{3}-\frac{1}{9}$ ladder (residuals $\lesssim 10^{-81}$),
- the sanity values (4) = -1 and (5) = $\ln 4$ to 30 digits.

Notes

- The main derivation (Sections 0–6) is complete and self-contained: the only tools are the fundamental theorem of calculus, Tonelli’s theorem for nonnegative integrands (twice: once for the s -integration, once for termwise series integration via monotone convergence), elementary partial fractions, and the definition $\text{Li}_2(z) = -\int_0^z \ln(1-w) dw/w$. Remarkably, the primary form $-6 \ln^2 2 - 3 \text{Li}_2(1/4)$ requires no dilogarithm functional equation at all — the $\text{Li}_2(1/2)$ contributions of the two pieces in Step 5 cancel.
- The equivalent forms in Section 7 use Euler’s $\zeta(2) = \pi^2/6$, which I cite as classical (Basel problem) rather than re-derive; every dilogarithm functional equation used is proved by differentiation from scratch.
- Section 9 (about the indefinite integral (6)) is an honest remark, not a theorem: I do not prove that $\int e^{2x} \text{Ei}(x) dx/x$ is not expressible via elementary functions and Ei; deciding that would require differential-algebra machinery (Liouville-type theory for Ei-extensions) beyond the scope here. The requested definite integral (7) is fully solved.
- The same scheme for the fourth power gives $\int_0^\infty E_1^4 = 4 \int_{[1,\infty)^2} \frac{\ln(2+u+v)}{uv(1+u+v)} du dv$, a weight-3 (trilogarithm-level) object; the method generalizes but the answers leave the dilogarithm class.