Cleo Bench Problem 2

Crazy $\int_0^\infty {}_3F_2\left(\begin{matrix}\frac58,\frac58,\frac98\\ \frac12,\frac{13}{8}\end{matrix}\,\middle|\,-x\right)^2\frac{dx}{\sqrt{x}}$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^\infty {}_3F_2\left(\begin{matrix}\frac58,\ \frac58,\ \frac98\\ \frac12,\ \frac{13}{8}\end{matrix}\,\middle|\,-x\right)^2\frac{dx}{\sqrt{x}}.$$

Result

$$\boxed{I = \frac{25\,\Gamma\left(\frac34\right)^2}{3\sqrt{\pi}}\left(2\ln(1+\sqrt2)-\sqrt2\right) = \frac{50\,\pi^{3/2}}{3\,\Gamma\left(\frac14\right)^2}\left(2\ln(1+\sqrt2)-\sqrt2\right)}$$

$$I = 2.4606853024494251089819676790449371813495806631514747 81\ldots$$

(The two forms agree because $\Gamma(\frac14)\Gamma(\frac34)=\pi\sqrt2$, hence $\Gamma(\frac34)^2 = 2\pi^2/\Gamma(\frac14)^2$. Also $2\ln(1+\sqrt2) = \ln(3+2\sqrt2) = 2\,\operatorname{arcsinh}1$.)

Derivation

Throughout put

$$a=\tfrac58,\qquad p=2a=\tfrac54.$$

The parameter pattern of the hypergeometric is exactly

$${}_3F_2\left(\begin{matrix}a,\ a,\ a+\frac12\\ \frac12,\ a+1\end{matrix}\,\middle|\,-x\right) =: F(x),$$

since $\frac58=a$, $\frac98=a+\frac12$, $\frac{13}{8}=a+1$. This pattern is what makes the problem tractable: the pair $\left(a,a+\frac12;\frac12\right)$ triggers a quadratic (elementary) ${}_2F_1$ evaluation, and the pair $\left(a;a+1\right)$ is a single Euler-type integration.

All powers of complex numbers below are **principal branches**; every complex number raised to a power will lie in the open right half-plane $\Re z>0$ (or be a positive real), so no branch ambiguity ever arises.

*Problem originally posed on Mathematics Stack Exchange (question 712798, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Step 1. An elementary form for the ${}_2F_1$ kernel

Claim. For all real s,

$$ {}_2F_1\Big(a,\ a+\tfrac12;\ \tfrac12;\ -s^2\Big) = \frac{(1+is)^{-2a}+(1-is)^{-2a}}{2} = \Re\,(1+is)^{-2a}. \tag{1} $$

Proof. For $|z|<1$ the binomial series gives

$$ \frac{(1-z)^{-2a}+(1+z)^{-2a}}{2} = \sum_{m\ \text{even}} \frac{(2a)_m}{m!}\,z^m = \sum_{k\ge0}\frac{(2a)_{2k}}{(2k)!}\,z^{2k}. $$

Splitting the products over even/odd factors (the Pochhammer duplication identities)

$$ (2a)_{2k} = 4^k\,(a)_k\Big(a+\tfrac12\Big)_k, \qquad (2k)! = 4^k\,k!\,\Big(\tfrac12\Big)_k, $$

we get $\sum_k \frac{(a)_k(a+\frac12)_k}{(\frac12)_k\,k!} z^{2k} = {}_2F_1(a, a+\frac12; \frac12; z^2)$ for $|z|<1$.

Now fix the branch discussion for $z = is$, $s\in\mathbb{R}$. The left side, as a function of s, is $\varphi(s^2)$ where $\varphi(w) = {}_2F_1(a, a+\frac12; \frac12; -w)$ is analytic on $\mathbb{C}\setminus(-\infty,-1]$; since $-s^2\le 0$ we stay in its domain, so $s\mapsto\varphi(s^2)$ is real-analytic on all of $\mathbb{R}$. The right side of (1) is real-analytic on $\mathbb{R}$ as well, because $1\pm is$ lies in the closed right half-plane minus 0 for every real s, away from the cut of the principal power. Both sides agree for $|s|<1$ by the series computation, hence for all real s by analytic continuation. Finally $\overline{(1+is)^{-2a}} = (1-is)^{-2a}$ for real s, so the average is $\Re(1+is)^{-2a}$. □

Step 2. Euler integral for the lower-parameter shift $a\mapsto a+1$

Since $\dfrac{(a)_n}{(a+1)_n} = \dfrac{a}{a+n} = a\displaystyle\int_0^1 t^{\,a+n-1}\,dt$, termwise integration of the (uniformly convergent) series for $0\le x<1$ gives

$$ F(x) = a\int_0^1 t^{\,a-1}\,{}_2F_1\Big(a,\ a+\tfrac12;\ \tfrac12;\ -xt\Big)\,dt. $$

Both sides are analytic in x on $\Omega = \mathbb{C}\setminus(-\infty,-1]$ — the left side is the principal branch of the ${}_3F_2$; the right side is analytic on Ω because for $x\in\Omega$ and $t\in(0,1]$ we have $xt\in\Omega$ (the product xt can only land on $(-\infty,-1]$ if x itself is real ≤ -1), the integrand is jointly continuous and locally uniformly bounded, so Morera/dominated convergence applies. Agreement on $[0,1)$ therefore extends to all $x\ge0$.

Inserting (1) with $s=\sqrt{xt}$ and substituting $t=v^2$ ($dt = 2v\,dv$):

$$ F(x) = 2a\int_0^1 v^{\,2a-1}\,\Re\,(1+iv\sqrt{x}\,)^{-2a}\,dv, \qquad x\ge 0. \tag{2} $$

(Representation (2) was verified numerically to 50 digits at $x = 0.7,\ 3.3,\ 100$; see the verification section.)

Step 3. Square, substitute $x=y^2$, and interchange (Fubini)

With $x=y^2$, $dx/\sqrt{x} = 2\,dy$, and writing the square of (2) as a double integral,

$$ I = 2\int_0^\infty F(y^2)^2\,dy = 2\,(2a)^2\int_0^\infty\iint_{[0,1]^2}(uv)^{2a-1}\,\Re(1+iuy)^{-p}\,\Re(1+ivy)^{-p}\,du\,dv\,dy. $$

To swap the y-integral with the (u,v)-integral we check absolute integrability. Since $|\Re(1+iuy)^{-p}| \le |1+iuy|^{-p} = (1+u^2y^2)^{-p/2} \le 1$, with $m = \max(u,v)$,

$$\int_0^\infty |\Re(1+iuy)^{-p}\,\Re(1+ivy)^{-p}|\,dy \le \int_0^\infty (1+m^2y^2)^{-p/2}\,dy = \frac{c_1}{m}, \qquad c_1 = \int_0^\infty \frac{ds}{(1+s^2)^{5/8}} < \infty$$

(c_1 converges because $2\cdot\frac{5}{8} = \frac{5}{4} > 1$), and

$$\iint_{[0,1]^2} (uv)^{1/4}\,\frac{c_1}{\max(u,v)}\,du\,dv = 2c_1\int_0^1 u^{-3/4}\Big(\int_0^u v^{1/4}dv\Big)du = \frac{16\,c_1}{15} < \infty.$$

(This bound also proves $I < \infty$.) Hence, by Fubini,

$$I = 2\,(2a)^2 \iint_{[0,1]^2} (uv)^{2a-1}\,W(u,v)\,du\,dv, \qquad W(u,v) = \int_0^\infty \Re(1+iuy)^{-p}\,\Re(1+ivy)^{-p}\,dy. \tag{3}$$

Step 4. The kernel integral $W(u,v)$ in closed form

Write $A = (1+iuy)^{-p}$, $B = (1+ivy)^{-p}$; for real y, $\bar B = (1-ivy)^{-p}$, and

$$\Re A\,\Re B = \tfrac12\Re(AB) + \tfrac12\Re(A\bar B) \implies W = \tfrac12\,\Re J_+ + \tfrac12\,\Re J_-,$$

$$J_+ = \int_0^\infty (1+iuy)^{-p}(1+ivy)^{-p}\,dy, \qquad J_- = \int_0^\infty (1+iuy)^{-p}(1-ivy)^{-p}\,dy.$$

(a) $\Re J_+ = 0$. For $\kappa > 0$ the principal power $(1+i\kappa y)^{-p}$ is analytic in y except on the cut $\{y = it:\ t \ge 1/\kappa\}$ (where $1+i\kappa y \in (-\infty,0]$), which lies on the **positive** imaginary axis. So the integrand of J_+ is analytic on the closed fourth quadrant. On the arc $|y| = R$, $\arg y \in [-\frac{\pi}{2}, 0]$, the integrand is $O(R^{-2p}) = O(R^{-5/2})$, so the arc contributes $O(R^{-3/2}) \to 0$. By Cauchy's theorem we may rotate the ray of integration to the negative imaginary axis, $y = -i\tau$:

$$J_+ = -i\int_0^\infty (1+u\tau)^{-p}(1+v\tau)^{-p}\,d\tau \ \in\ i\,\mathbb{R} \quad\implies\quad \Re J_+ = 0.$$

(b) $\Re J_-$. For $y \ge 0$ both $1+iuy$ and $1-ivy$ lie in the open right half-plane, where principal powers multiply: $(z_1z_2)^{-p} = z_1^{-p}z_2^{-p}$ whenever $|\arg z_1|, |\arg z_2| < \frac{\pi}{2}$. Completing the square,

$$(1+iuy)(1-ivy) = 1+uvy^2+i(u-v)y = uv\Big[(y+i\delta)^2+\sigma^2\Big], \qquad \delta = \frac{u-v}{2uv}, \quad \sigma = \frac{u+v}{2uv},$$

which is consistent at $y=0$ because $uv(\sigma^2-\delta^2) = \frac{(u+v)^2-(u-v)^2}{4uv} = 1$; note $\sigma > |\delta|$. Since $\Re\big[(y+i\delta)^2+\sigma^2\big] = \frac{1+uvy^2}{uv} > 0$ and $uv > 0$, principal branches again factor, and

$$J_- = (uv)^{-p}\int_\Gamma (w^2+\sigma^2)^{-p}\,dw,$$

where Γ is the horizontal ray from $i\delta$ to $i\delta+\infty$ ($w = y+i\delta$). The principal power $(w^2+\sigma^2)^{-p}$ is analytic off $\{w = it:\ |t| \ge \sigma\}$, and the strip $|\Im w| \le |\delta| < \sigma$, $\Re w \ge 0$, avoids that set. Deform Γ to the vertical segment from $i\delta$ down to 0 followed by $[0,\infty)$; the far vertical connector at $\Re w = R$ contributes $O(R^{-5/2})\cdot|\delta| \to 0$. On the segment $w = i\tau$ (τ between 0 and δ) we have $w^2+\sigma^2 = \sigma^2-\tau^2 \in (0,\sigma^2]$, so the integrand is **real** while $dw = i\,d\tau$ — that piece is purely imaginary. Hence

$$\Re J_- = (uv)^{-p}\int_0^\infty (y^2+\sigma^2)^{-p}dy = (uv)^{-p}\,\sigma^{1-2p}\,\frac{\sqrt{\pi}\,\Gamma\big(p-\frac12\big)}{2\,\Gamma(p)} = \frac{\sqrt{\pi}\,\Gamma\big(p-\frac12\big)}{2\,\Gamma(p)}\,\frac{2^{2p-1}\,(uv)^{p-1}}{(u+v)^{2p-1}},$$

using the Beta evaluation $\int_0^\infty (y^2+\sigma^2)^{-p}dy = \frac{1}{2}\sigma^{1-2p}B(\frac{1}{2}, p-\frac{1}{2})$ (substitute $y = \sigma\tan\theta$), valid since $p = \frac{5}{4} > \frac{1}{2}$.

So

$$W(u,v) = \frac{\sqrt{\pi}\,\Gamma\left(p-\frac{1}{2}\right)}{4\,\Gamma(p)}\,\frac{2^{2p-1}(uv)^{p-1}}{(u+v)^{2p-1}}. \tag{4}$$

(Identity (4) was verified numerically to ~ 50 digits at several (u,v); see below.)

Step 5. Assembling: an elementary double integral

Insert (4) into (3). With $a = \frac{5}{8}$, $p = \frac{5}{4}$ we have $(uv)^{2a-1}(uv)^{p-1} = (uv)^{1/2}$ and $2p-1 = \frac{3}{2}$:

$$I = (2a)^2\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma\left(\frac{5}{4}\right)}\,2^{3/2}\,D, \qquad D = \iint_{[0,1]^2} \frac{\sqrt{uv}}{(u+v)^{3/2}}\,du\,dv.$$

By the $u \leftrightarrow v$ symmetry and the scaling $v = ut$ on the region $v < u$,

$$D = 2\int_0^1\int_0^u \frac{\sqrt{uv}}{(u+v)^{3/2}}\,dv\,du = 2\int_0^1 u^{1/2}\,du \int_0^1 \frac{\sqrt{t}}{(1+t)^{3/2}}\,dt = \frac{4}{3}\int_0^1 \frac{\sqrt{t}\,dt}{(1+t)^{3/2}}.$$

Substituting $t = \sinh^2 w$ turns the last integral into $2\int \tanh^2 w\,dw = 2(w - \tanh w)$, so

$$\int_0^1 \frac{\sqrt{t}\,dt}{(1+t)^{3/2}} = 2\left(\operatorname{arcsinh} 1 - \frac{1}{\sqrt{2}}\right) = 2\ln(1+\sqrt{2}) - \sqrt{2}, \qquad D = \frac{4}{3}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right).$$

Step 6. Final constants

$$I = \frac{25}{16}\cdot\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma\left(\frac{5}{4}\right)}\cdot 2^{3/2}\cdot\frac{4}{3}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right) = \frac{25\cdot 2^{7/2}}{96}\cdot\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right).$$

Now $\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$ and the reflection formula $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}$ give $\frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(5/4)} = \frac{4\sqrt{2}\,\pi^{3/2}}{\Gamma(1/4)^2}$, hence

$$I = \frac{25\cdot 2^{7/2}\cdot 4\sqrt{2}}{96}\cdot\frac{\pi^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right) = \frac{50\,\pi^{3/2}}{3\,\Gamma\left(\frac{1}{4}\right)^2}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right),$$

and equivalently, using $\Gamma(\frac{3}{4})^2 = 2\pi^2/\Gamma(\frac{1}{4})^2$,

$$I = \frac{25\,\Gamma\left(\frac{3}{4}\right)^2}{3\sqrt{\pi}}\left(2\ln(1+\sqrt{2}) - \sqrt{2}\right).$$

■

Remark. The same computation evaluates the whole family $\int_0^\infty {}_3F_2(a,a,a+\frac{1}{2};\frac{1}{2},a+1;-x)^2 x^{-1/2}dx$ for $\frac{1}{2} < a < 1$ (so that $p = 2a$ satisfies $p > \frac{1}{2}$ for Step 4 and the Fubini bounds hold); the double integral $\iint (uv)^{3a-2}(u+v)^{1-4a}\,du\,dv$ is again elementary-hypergeometric. Wait — for general a the exponents are $(uv)^{2a-1+p-1} = (uv)^{3a-2}$ and $(u+v)^{4a-1}$; the special value $a = \frac{5}{8}$ makes it the pleasant $\iint \sqrt{uv}\,(u+v)^{-3/2}$. *(This remark is illustrative only; the boxed result for $a = 5/8$ is what was proved above.)*

Numerical verification

All computations with mpmath. Working precision `mp.dps = 60` (spot checks at 50).

1. **Representation (2)** vs `mp.hyp3f2(5/8,5/8,9/8,1/2,13/8,-x)` at $x = 0.7$, 3.3, 100: differences 0, 0, 8×10^{-53} at 50 dps.

2. **Kernel identity (4)** vs direct quadrature of $W(u,v) = \int_0^\infty \Re(1+iuy)^{-5/4}\Re(1+ivy)^{-5/4}dy$ at $(u,v) = (0.3, 0.9)$, $(0.5, 0.5)$, $(0.05, 0.77)$: differences $\lesssim 3 \times 10^{-51}$.

3. **Direct evaluation of** I at 60 dps using `mp.hyp3f2` itself (no ingredient of the derivation): $I = \int_0^1 F(x)^2 x^{-1/2}dx + \int_0^1 F(1/w)^2 w^{-3/2}dw$, computed as $2\int_0^1 F(y^2)^2 dy$ (smooth) plus, with $w = z^4$, $4\int_0^1 F(z^{-4})^2 z^{-3}dz$ (integrand $\sim z^2 \log^2 z$ at 0), each by tanh–sinh quadrature (`maxdegree=9`):

$$I_{\text{num}} = 2.46068530244942510898196767904493718134958066315147478 1\ldots$$

4. **Closed form** at 60 dps:

$$\frac{25\,\Gamma(3/4)^2}{3\sqrt{\pi}}(2\ln(1+\sqrt{2}) - \sqrt{2})$$
$$= 2.46068530244942510898196767904493718134958066315147478 1\ldots$$

   Relative difference $|I_{\text{num}} - I_{\text{cf}}|/I_{\text{cf}} \approx 1.3 \times 10^{-61}$: **60 significant digits agree** (the full working precision).

5. **Independent cross-check (Mellin–Parseval).** With $\tilde{F}(s) = \frac{\Gamma(s)\Gamma(\frac{5}{8}-s)^2\Gamma(\frac{9}{8}-s)\,\Gamma(\frac{1}{2})\Gamma(\frac{13}{8})}{\Gamma(\frac{5}{8})^2\Gamma(\frac{9}{8})\,\Gamma(\frac{1}{2}-s)\Gamma(\frac{13}{8}-s)}$ (the Mellin transform of F), Parseval gives $I = \frac{1}{\pi}\int_0^\infty \Re[\tilde{F}(\frac{1}{4}+it)\tilde{F}(\frac{1}{4}-it)]dt$; after the cancellation of $\Gamma(s)$ and $\Gamma(\frac{1}{2}-s)$ in the product this Gamma-integral was evaluated numerically at 60 dps and agrees with the closed form to 6×10^{-61}. This route uses none of Steps 1–6.

Scripts: `step1_checks.py`, `step2_direct.py`, `step3_highprec.py`, `step4_digits.py` in the scratch work directory.

Notes

- The proof is self-contained: the only inputs are the binomial series, the Pochhammer duplication identities, Cauchy's theorem (two explicitly justified contour deformations, with the decay rate $|y|^{-5/2}$ making all arcs vanish), the Beta integral $\int_0^\infty (y^2+\sigma^2)^{-p}dy$, and one Fubini interchange with an explicit absolutely-convergent majorant (which simultaneously proves $I < \infty$).

- Branch handling: every principal power occurs at an argument in the closed right half-plane off the cut, and both factorization steps $(z_1 z_2)^{-p} = z_1^{-p} z_2^{-p}$ are applied only when $|\arg z_i| < \pi/2$, where they are valid.

- The analytic-continuation arguments in Steps 1–2 (from $|x| < 1$ to $x \geq 0$) are standard and fully justified (domains are connected, both sides analytic, agreement on a set with a limit point); they were additionally confirmed numerically.

- The Mellin–Parseval computation in the verification section is only a cross-check, not part of the proof; its own hypotheses (strip of analyticity $0 < \Re s < \frac{5}{8}$ containing $\Re s = \frac{1}{4}$, L^2/L^1 conditions from the $e^{-2\pi|t|}$ decay of the Gamma product) do hold, but the derivation does not rely on it.
- No step is conjectural; I consider the result proved.