Cleo Bench Problem 22

Derivative of the Meijer G-function with respect to one of its parameters

Derivation by Claude (Fable 5), closed-book*

July 2026

Problem

Evaluate in closed form the derivative of the Meijer G-function with respect to one of its upper parameters,

$$\mathcal{D} = \partial_\alpha \, G^{2,1}_{2,3}\!\left(1 \,\middle|\, \begin{matrix} 1, \alpha \\ 1, 1, 0 \end{matrix}\right)\Bigg|_{\alpha=1}.$$

Here the Meijer G-function is taken in the standard (MathWorld / Wolfram / mpmath) convention

$$G^{m,n}_{p,q}\!\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\,\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\,\prod_{j=n+1}^{p}\Gamma(a_j - s)}\, z^s \, ds,$$

with L a contour separating the poles of the factors $\Gamma(b_j - s)$, $j \le m$, from those of the factors $\Gamma(1 - a_j + s)$, $j \le n$.

Result

$$\boxed{\begin{aligned} \mathcal{D} &= \gamma + E_1(1) = \gamma - \mathrm{Ei}(-1) = \mathrm{Ein}(1) \\ &= \int_0^1 \frac{1 - e^{-t}}{t}\, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} \\ &= 0.79659959929705313428367586554252408\dots \end{aligned}}$$

where γ is the Euler–Mascheroni constant, $E_1(x) = \int_x^\infty t^{-1}e^{-t}dt$, Ei is the exponential integral, and $\mathrm{Ein}(z) = \int_0^z t^{-1}(1 - e^{-t})\, dt$ is the complementary (entire) exponential integral.

As a by-product, the same method gives the value of the function itself: $G^{2,1}_{2,3}\!\left(1 \,\middle|\, {}^{1,1}_{1,1,0}\right) = 1 - e^{-1}$.

Derivation

1. The Mellin–Barnes representation and its contour

For our indices $m = 2,\ n = 1,\ p = 2,\ q = 3$ and parameters $a_1 = 1,\ a_2 = \alpha,\ b_1 = b_2 = 1,\ b_3 = 0$, the definition reads

$$G(\alpha) := G^{2,1}_{2,3}\!\left(1 \,\middle|\, \begin{matrix} 1, \alpha \\ 1, 1, 0 \end{matrix}\right) = \frac{1}{2\pi i}\int_L \Phi(s,\alpha)\, ds, \qquad \Phi(s,\alpha) = \frac{\Gamma(1-s)^2\,\Gamma(s)}{\Gamma(1+s)\,\Gamma(\alpha - s)}. \tag{1}$$

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[*] Problem originally posed on Mathematics Stack Exchange (question 577849, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Indeed: the group $j \le m = 2$ contributes $\Gamma(b_1 - s)\Gamma(b_2 - s) = \Gamma(1-s)^2$; the group $j \le n = 1$ contributes $\Gamma(1 - a_1 + s) = \Gamma(s)$; the denominator gets $\Gamma(1 - b_3 + s) = \Gamma(1+s)$ from $j = m+1 = 3$ and $\Gamma(a_2 - s) = \Gamma(\alpha - s)$ from $j = n+1 = 2$; and $z^s = 1^s = 1$.

Contour. The poles of $\Gamma(1-s)^2$ lie at $s = 1, 2, 3, \dots$ (double poles), and the poles of $\Gamma(s)$ lie at $s = 0, -1, -2, \dots$. The factor $1/\Gamma(\alpha - s)$ is entire in s (reciprocal Gamma is entire), so it contributes no poles. Hence for every α any vertical line $L = \{\Re s = c\}$ with $0 < c < 1$, oriented upward, is an admissible separating contour; we fix $c = \frac{1}{2}$.

Convergence. We use the classical uniform asymptotics on vertical lines (uniform version of Stirling's formula; e.g. NIST DLMF 5.11.9): for σ in a compact set,

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}\, |t|^{\sigma - \frac{1}{2}} e^{-\pi|t|/2}\bigl(1 + O(|t|^{-1})\bigr), \qquad |t| \to \infty, \tag{2}$$

and $\psi(\sigma + it) = \log|t| + O(1)$ (from $\psi(w) = \log w + O(1/|w|)$, valid uniformly in $|\arg w| \le \pi - \delta$).

For $s = \frac{1}{2} + it$, (2) gives (with α ranging over the closed disk $D_{1/4} = \{|\alpha - 1| \le \frac{1}{4}\}$):

- numerator: $|\Gamma(1-s)|^2 |\Gamma(s)| \le C_1(1 + |t|)^{1/2 - c} e^{-3\pi|t|/2}$, since the $|t|$-exponents are $(1-c) - \frac{1}{2}$ (twice) and $c - \frac{1}{2}$;
- $1/|\Gamma(1+s)| \le C_2(1 + |t|)^{-c-1/2} e^{\pi|t|/2}$;
- $1/|\Gamma(\alpha - s)| \le C_3(1 + |t|)^{1/4} e^{\pi|t|/2}$ uniformly for $\alpha \in D_{1/4}$. Indeed $w = \alpha - s$ has $\Re w = \Re\alpha - \frac{1}{2} \in [\frac{1}{4}, \frac{3}{4}]$ and $|\Im w| \in [|t| - \frac{1}{4},\ |t| + \frac{1}{4}]$; by (2), $|\Gamma(w)| \ge c_0(1 + |t|)^{\Re w - 1/2} e^{-\pi(|t| + 1/4)/2}$ for $|t| \ge T_0$, while on the compact set $\{|t| \le T_0,\ \alpha \in D_{1/4}\}$ the continuous non-vanishing function $\Gamma(\alpha - s)$ is bounded away from 0 (Γ has no zeros, and its poles lie outside the strip $\frac{1}{4} \le \Re w \le \frac{3}{4}$).

Multiplying,

$$|\Phi(\tfrac{1}{2} + it, \alpha)| \;\le\; C\,(1 + |t|)^{-3/4}\, e^{-\pi|t|/2} \qquad \text{for all } t \in \mathbb{R},\ \alpha \in D_{1/4}. \tag{3}$$

So the integral (1) converges absolutely, uniformly for $\alpha \in D_{1/4}$. (The exponential decay rate $e^{-\pi|t|/2}$ is exactly the general rule $e^{-(m+n-\frac{p+q}{2})\pi|t|}$ with $m + n - \frac{p+q}{2} = \frac{1}{2} > 0$, which is the condition for the G-function to exist at $|z| = 1$, $\arg z = 0$.) By Cauchy's theorem the value is independent of $c \in (0,1)$: shifting the line crosses no poles and the horizontal connecting segments vanish by (3)-type bounds, which hold uniformly for c in compact subsets of $(0,1)$.

(Consistency of conventions was additionally confirmed numerically: the integral (1) reproduces mpmath's `meijerg` to about 60 significant digits for $\alpha \in \{1,\ 1.3,\ 0.6,\ 1.2 + 0.3i\}$; see §Numerical verification.)

2. Analyticity in α and differentiation under the integral sign

For each fixed s on L, $\alpha \mapsto \Phi(s, \alpha)$ is entire (reciprocal Gamma is entire). By the uniform bound (3):

- $G(\alpha)$ is continuous on $D_{1/4}$ (dominated convergence);
- for every closed triangle $\Delta \subset \operatorname{int} D_{1/4}$, Fubini (justified by (3), a triangle has finite length) gives $\oint_\Delta G(\alpha)\, d\alpha = \frac{1}{2\pi i} \int_L \oint_\Delta \Phi(s, \alpha)\, d\alpha\, ds = 0$, so G is analytic in $\operatorname{int} D_{1/4}$ by Morera's theorem.

By the Cauchy integral formula for the derivative and Fubini once more (again legitimate by (3), the circle $|\alpha-1|=\frac{1}{8}$ being compact),

$$G'(1)=\frac{1}{2\pi i}\oint_{|\alpha-1|=\frac{1}{8}}\frac{G(\alpha)}{(\alpha-1)^2}\,d\alpha$$
$$=\frac{1}{2\pi i}\int_L\left[\frac{1}{2\pi i}\oint_{|\alpha-1|=\frac{1}{8}}\frac{\Phi(s,\alpha)}{(\alpha-1)^2}\,d\alpha\right]ds=\frac{1}{2\pi i}\int_L\partial_\alpha\Phi(s,\alpha)\Big|_{\alpha=1}\,ds.$$

Since $\partial_\alpha\,\Gamma(\alpha-s)^{-1}=-\psi(\alpha-s)/\Gamma(\alpha-s)$, at $\alpha=1$:

$$\partial_\alpha\Phi(s,\alpha)\Big|_{\alpha=1}=-\frac{\Gamma(1-s)^2\Gamma(s)\,\psi(1-s)}{\Gamma(1+s)\,\Gamma(1-s)}=-\frac{\Gamma(1-s)\,\psi(1-s)}{s},$$

using $\Gamma(s)/\Gamma(1+s)=1/s$. Hence

$$\mathcal{D}=-\frac{1}{2\pi i}\int_{(\frac{1}{2})}\frac{\Gamma(1-s)\,\psi(1-s)}{s}\,ds. \tag{4}$$

(The differentiated integrand again obeys a bound $C(1+|t|)^{-1}\log(2+|t|)\,e^{-\pi|t|/2}$ by (2), so (4) converges absolutely.)

3. Reflection of the variable

Substitute $u=1-s$. As s runs up the line $\Re s=\frac{1}{2}$, u runs **down** the line $\Re u=\frac{1}{2}$, and $ds=-du$; the two sign changes cancel, so with the standard upward orientation

$$\mathcal{D}=-\frac{1}{2\pi i}\int_{(\frac{1}{2})}\frac{\Gamma(u)\,\psi(u)}{1-u}\,du=-\frac{1}{2\pi i}\int_{(\frac{1}{2})}\frac{\Gamma'(u)}{1-u}\,du, \tag{5}$$

since $\Gamma'(u)=\Gamma(u)\psi(u)$.

4. A Mellin pair for Γ'

Lemma. For $0<c'<1$ and every $y>0$,

$$\frac{1}{2\pi i}\int_{(c')}\Gamma'(u)\,y^{-u}\,du\;=\;e^{-y}\ln y. \tag{6}$$

Proof. Euler's integral $\Gamma(u)=\int_0^\infty e^{-y}y^{u-1}dy$ converges absolutely and locally uniformly in $\Re u>0$, so it defines an analytic function there, and differentiation under the integral sign (dominated convergence: for $0<\sigma_1\le\Re u\le\sigma_2$, $|e^{-y}y^{u-1}\ln y|\le e^{-y}\max(y^{\sigma_1-1},y^{\sigma_2-1})|\ln y|\in L^1(0,\infty)$) gives

$$\Gamma'(u)=\int_0^\infty e^{-y}\,(\ln y)\,y^{u-1}\,dy,\qquad \Re u>0;$$

that is, Γ' is the Mellin transform of $g(y):=e^{-y}\ln y$ on the strip $\Re u>0$.

Substituting $y=e^{-w}$ turns the Mellin transform on the line $\Re u=c'$ into a Fourier transform:

$$\Gamma'(c'+i\tau)=\int_{\mathbb{R}}h(w)\,e^{-i\tau w}\,dw=\widehat{h}(\tau),\qquad h(w):=g(e^{-w})e^{-c'w}=-\,w\,e^{-e^{-w}}e^{-c'w}.$$

Now h is continuous on $\mathbb{R}$ and $h\in L^1(\mathbb{R})$: as $w\to+\infty$, $|h(w)|\le|w|e^{-c'w}$ decays exponentially; as $w\to-\infty$, the factor $e^{-e^{-w}}$ decays double-exponentially and beats $|w|e^{-c'w}$. Moreover $\widehat{h}(\tau)=\Gamma'(c'+i\tau)=\Gamma(c'+i\tau)\psi(c'+i\tau)$ satisfies, by (2) and $\psi(c'+i\tau)=O(\log|\tau|)$,

$$|\widehat{h}(\tau)|\le C(1+|\tau|)^{c'-1/2}\log(2+|\tau|)\,e^{-\pi|\tau|/2},$$

so $\widehat{h} \in L^1(\mathbb{R})$. By the Fourier inversion theorem (valid pointwise when $h \in L^1 \cap C^0$ and $\widehat{h} \in L^1$),

$$h(w) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{h}(\tau) e^{i\tau w}\, d\tau \qquad \text{for every } w \in \mathbb{R}.$$

With $y = e^{-w}$, i.e. $y^{-i\tau} = e^{i\tau w}$,

$$\frac{1}{2\pi i} \int_{(c')} \Gamma'(u) y^{-u} du = \frac{y^{-c'}}{2\pi} \int_{\mathbb{R}} \widehat{h}(\tau) e^{i\tau w} d\tau = y^{-c'} h(w) = y^{-c'} \cdot g(y)\, y^{c'} = e^{-y} \ln y. \qquad \blacksquare$$

5. Fubini and reduction to a one-dimensional integral

On the line $\Re u = c' = \frac{1}{2} < 1$ we have the absolutely convergent elementary representation

$$\frac{1}{1-u} = \int_1^{\infty} x^{u-2}\, dx.$$

Insert this into (5). The double integral converges absolutely:

$$\int_{\mathbb{R}} \int_1^{\infty} |\Gamma'(\tfrac{1}{2} + i\tau)|\, x^{-3/2}\, dx\, d\tau = \Big(\int_{\mathbb{R}} |\Gamma'(\tfrac{1}{2} + i\tau)| d\tau\Big) \cdot 2 < \infty,$$

by the exponential decay of Γ' on the line. Hence Fubini's theorem applies and, using the Lemma with $y = 1/x \in (0, 1]$ (note $x^u = y^{-u}$):

$$\mathcal{D} = -\int_1^{\infty} \frac{1}{x^2} \left[\frac{1}{2\pi i} \int_{(\frac{1}{2})} \Gamma'(u)\, x^u\, du\right] dx = -\int_1^{\infty} \frac{e^{-1/x}\, \ln(1/x)}{x^2}\, dx.$$

Substituting $t = 1/x$ ($dt = -x^{-2}dx$):

$$\mathcal{D} = -\int_0^1 e^{-t}\, \ln t\; dt. \tag{7}$$

6. Evaluation of the elementary integral

(a) Closed form via γ and E_1. Differentiating Euler's integral at $u = 1$ (justified in §4) and using $\Gamma'(1) = \psi(1)\Gamma(1) = \psi(1)$:

$$\int_0^{\infty} e^{-t} \ln t\, dt = \Gamma'(1) = \psi(1) = -\gamma.$$

(The classical fact $\psi(1) = -\gamma$ follows from the Weierstrass product $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n \geq 1}(1 + \frac{z}{n}) e^{-z/n}$: logarithmic differentiation gives $\psi(z) = -\frac{1}{z} - \gamma + \sum_{n \geq 1}(\frac{1}{n} - \frac{1}{n+z})$, and at $z = 1$ the sum telescopes to 1, so $\psi(1) = -1 - \gamma + 1 = -\gamma$.)

Also, integrating by parts on $[1, \infty)$ (boundary terms vanish since $\ln 1 = 0$ and $e^{-t} \ln t \to 0$),

$$\int_1^{\infty} e^{-t} \ln t\, dt = [-e^{-t} \ln t]_1^{\infty} + \int_1^{\infty} \frac{e^{-t}}{t}\, dt = E_1(1).$$

Therefore

$$\mathcal{D} = -\int_0^1 e^{-t} \ln t\, dt = -\int_0^{\infty} e^{-t} \ln t\, dt + \int_1^{\infty} e^{-t} \ln t\, dt = \gamma + E_1(1) = \gamma - \mathrm{Ei}(-1),$$

using $E_1(x) = -\mathrm{Ei}(-x)$ for $x > 0$.

(b) Equivalent forms. Integrating (7) by parts with $v(t) = 1 - e^{-t}$ (the antiderivative of e^{-t} vanishing at 0, so that the boundary term at 0 dies: $(1 - e^{-t})\ln t \sim t \ln t \to 0$):

$$\mathcal{D} = -\big[(1 - e^{-t})\ln t\big]_0^1 + \int_0^1 \frac{1 - e^{-t}}{t}\,dt = \mathrm{Ein}(1),$$

and termwise integration of the uniformly convergent series $\frac{1-e^{-t}}{t} = \sum_{n\geq 1} \frac{(-1)^{n-1} t^{n-1}}{n!}$ on $[0,1]$ gives

$$\mathcal{D} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} = 1 - \frac{1}{4} + \frac{1}{18} - \frac{1}{96} + \cdots .$$

(That $\mathrm{Ein}(1) = \gamma + E_1(1)$ is the classical connection formula $\mathrm{Ein}(z) = \gamma + \ln z + E_1(z)$ at $z = 1$, re-proved here by parts (a)–(b).)

7. Consistency checks inside the same framework

Residue expansion (independent route to the same series). In (5) one may close the contour to the left, around the double poles of $\Gamma'(u)$ at $u = -k$, $k \geq 0$; the arcs vanish by standard estimates for Mellin–Barnes integrals (reflection formula plus (2); the mechanism is identical to the classical Cahen–Mellin integral $\frac{1}{2\pi i}\int_{(c')} \Gamma(u) y^{-u} du = e^{-y}$). From $\Gamma(u) = \frac{(-1)^k}{k!}\big[\varepsilon^{-1} + \psi(k+1) + O(\varepsilon)\big]$ $(\varepsilon = u + k)$ we get $\Gamma'(u) = \frac{(-1)^k}{k!}\big[-\varepsilon^{-2} + O(1)\big]$, and with $\frac{1}{1-u} = \frac{1}{1+k}\big(1 + \frac{\varepsilon}{1+k} + O(\varepsilon^2)\big)$,

$$\operatorname*{Res}_{u=-k} \frac{\Gamma'(u)}{1-u} = -\frac{(-1)^k}{k!\,(k+1)^2}, \qquad \mathcal{D} = -\sum_{k\geq 0} \operatorname*{Res}_{u=-k} = \sum_{k\geq 0} \frac{(-1)^k}{k!\,(k+1)^2} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n \cdot n!},$$

in agreement with §6(b). (The main proof does **not** rely on this contour closing; it is a check.)

Value of the function itself. Exactly the same steps applied to $G(1)$ instead of $G'(1)$ — i.e. (1) at $\alpha = 1$, then $u = 1 - s$, giving $\frac{1}{2\pi i}\int_{(1/2)} \frac{\Gamma(u)}{1-u} du$, then Fubini and the Cahen–Mellin pair $\frac{1}{2\pi i}\int_{(c')} \Gamma(u) y^{-u} du = e^{-y}$ — yield

$$G^{2,1}_{2,3}\left(1 \,\middle|\, \begin{matrix} 1, 1 \\ 1, 1, 0 \end{matrix}\right) = \int_0^1 e^{-t}\,dt = 1 - e^{-1},$$

confirmed numerically to 40 digits, which also pins down the parameter conventions.

Numerical verification

All computations with mpmath at `mp.mp.dps = 60` (script `verify.py` in the scratch directory).

1. **Convention check.** The Mellin–Barnes integral (1) computed by direct quadrature along $\Re s = \frac{1}{2}$ (split points up to $|t| = 100$, integrand decay $e^{-\pi|t|/2}$) agrees with `mp.meijerg([[1],[alpha]], [[1,1],[0]], 1)` to $\sim 10^{-60}$ for $\alpha = 1,\ 1.3,\ 0.6,\ 1.2 + 0.3i$; and $G(1) = 0.63212055882855767840\ldots = 1 - e^{-1}$ (40 digits).

2. **Direct computation of $\mathcal{D}$** (two independent ways):

   - Cauchy-circle differentiation of mpmath's own `meijerg`: `mp.diff(G, 1, method='quad', radius=0.5)` gives

     $$\mathcal{D} = 0.79659959929705313428367586554252408007320662934683$$

     (imaginary part $\sim 10^{-72}$).

- Direct quadrature of the differentiated Mellin–Barnes integral (4):

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\Gamma(\tfrac{1}{2}-it)\,\psi(\tfrac{1}{2}-it)}{\tfrac{1}{2}+it}\,dt$$
$$= 0.796599599297053134283675865542524080073206629346 83.$$

3. **Closed form.** $\gamma + E_1(1) =$ `mp.euler + mp.e1(1)`

$$= 0.79659959929705313428367586554252408007320662934683;$$

the series $\sum_{n\geq 1}\frac{(-1)^{n-1}}{n\,n!}$ and $-\int_0^1 e^{-t}\ln t\,dt$ give the identical value.

4. **Agreement.** The `meijerg`-based derivative matches the closed form to all computed digits (relative difference 0 at 60-digit working precision; the line-integral route agrees to 61 digits). Conservatively: **at least 50 significant digits agree.**

Notes

- The proof is self-contained modulo standard textbook facts: uniform Stirling asymptotics on vertical strips (DLMF 5.11), $\psi(w) = \log w + O(1/w)$, Fubini–Tonelli, Morera + Cauchy integral formula for analyticity/differentiation under the integral, the Fourier inversion theorem for $h \in L^1 \cap C^0$ with $\widehat{h} \in L^1$, Euler's integral for Γ, and the Weierstrass product (for $\psi(1) = -\gamma$). Every interchange (differentiation under the integral in §2 and §4, Fubini in §5) is justified by the explicit absolute bounds (2)–(3).
- The contour-closing argument in §7 is presented only as an independent consistency check; the main chain (§§1–6) avoids arc estimates entirely.
- The identity $E_1(x) = -\operatorname{Ei}(-x)$ $(x > 0)$ is used only to restate the answer; the primary closed form is $\mathcal{D} = \gamma + E_1(1) = \operatorname{Ein}(1)$.
- Numerically, $\mathcal{D} = 0.796599599297053134283675865542524080073206629346833\ldots$