Cleo Bench Problem 18

Integral $\int_0^1 \frac{\ln\left(x+\sqrt{2}\right)}{\sqrt{2-x}\,\sqrt{1-x}\,\sqrt{x}}\mathrm{d}x$

Derivation by Claude (Fable 5), closed-book[*]

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Problem

Evaluate in closed form

$$I \;=\; \int_0^1 \frac{\ln\left(x+\sqrt{2}\right)}{\sqrt{(2-x)(1-x)\,x}}\,dx.$$

Result

$$\boxed{I \;=\; \frac{\Gamma\left(\frac{1}{4}\right)^2}{16\sqrt{2\pi}}\left(7\ln 2 \;+\; 4\ln(1+\sqrt{2}) \;-\; \pi\right)}$$

Equivalently, with $K := K\left(\frac{1}{\sqrt{2}}\right) = \dfrac{\Gamma(1/4)^2}{4\sqrt{\pi}}$ (lemniscatic complete elliptic integral, modulus convention $K(k)$),

$$I = \sqrt{2}\,K\;\ln C, \qquad C = 2^{7/8}\,(1+\sqrt{2})^{1/2}\,e^{-\pi/8} \;=\; 1.92416382125458199 87\ldots$$

Numerically $I = 1.716114368162902406441849005033005386647\ldots$

Derivation

Throughout, $k = \frac{1}{\sqrt{2}}$ (so $k' = \sqrt{1-k^2} = \frac{1}{\sqrt{2}} = k$), and $\theta_1, \theta_2, \theta_3, \theta_4$ are the Jacobi theta functions with nome q, $0 < q < 1$:

$$\theta_1(z) = 2\sum_{n\ge 0}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n{+}1)z, \quad \theta_2(z) = 2\sum_{n\ge 0} q^{(n+\frac{1}{2})^2}\cos(2n{+}1)z,$$

$$\theta_3(z) = 1 + 2\sum_{n\ge 1} q^{n^2}\cos 2nz, \quad \theta_4(z) = 1 + 2\sum_{n\ge 1}(-1)^n q^{n^2}\cos 2nz.$$

Write $t_j := \theta_j(0)$. It is convenient to use the two-sided forms

$$\theta_1(z) = -i\sum_{n\in\mathbb{Z}}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)iz}, \qquad \theta_2(z) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}e^{(2n+1)iz},$$

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[*] Problem originally posed on Mathematics Stack Exchange (question 570997, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

$$\theta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2inz}, \qquad \theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2inz}.$$

All are entire; for real $q \in (0,1)$ all Fourier coefficients are real, so $\overline{\theta_j(z)} = \theta_j(\bar{z})$.

We will also use the **Jacobi triple product** (classical; Jacobi 1829, see Whittaker–Watson §21.3):

$$\theta_1(z) = 2q^{1/4} \sin z \prod_{n\geq 1} (1-q^{2n})(1-q^{2n}e^{2iz})(1-q^{2n}e^{-2iz}),$$

$$\theta_2(z) = 2q^{1/4} \cos z \prod_{n\geq 1} (1-q^{2n})(1+q^{2n}e^{2iz})(1+q^{2n}e^{-2iz}),$$

$$\theta_3(z) = \prod_{n\geq 1} (1-q^{2n})(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz}),$$

$$\theta_4(z) = \prod_{n\geq 1} (1-q^{2n})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}).$$

Set $P := \prod_{n\geq 1}(1-q^{2n})$. From the products: for real z, $\theta_3(z) > 0$, $\theta_4(z) > 0$, and $\theta_1(z), \theta_2(z) > 0$ for $z \in (0, \frac{\pi}{2})$. The zero sets are: θ_1 vanishes exactly on $\pi\mathbb{Z} + \pi\tau\mathbb{Z}$, θ_2 on $\frac{\pi}{2} + \pi\mathbb{Z} + \pi\tau\mathbb{Z}$, θ_3 on $\frac{\pi}{2} + \frac{\pi\tau}{2} + \pi\mathbb{Z} + \pi\tau\mathbb{Z}$, θ_4 on $\frac{\pi\tau}{2} + \pi\mathbb{Z} + \pi\tau\mathbb{Z}$, where $q = e^{i\pi\tau}$ (all zeros simple; immediate from the products).

Step 0: quasi-periodicity and shift identities (elementary, from the series)

Directly from the two-sided series (reindexing $n \mapsto n \pm 1$ where needed), with $q = e^{i\pi\tau}$:

- **Periods.** $\theta_1(z+\pi) = -\theta_1(z)$, $\theta_2(z+\pi) = -\theta_2(z)$, $\theta_3(z+\pi) = \theta_3(z)$, $\theta_4(z+\pi) = \theta_4(z)$; and with $\mu(z) := q^{-1}e^{-2iz}$,

$$\theta_1(z+\pi\tau) = -\mu\,\theta_1(z), \quad \theta_2(z+\pi\tau) = \mu\,\theta_2(z),$$
$$\theta_3(z+\pi\tau) = \mu\,\theta_3(z), \quad \theta_4(z+\pi\tau) = -\mu\,\theta_4(z).$$

- **Quarter shifts in z:** $\theta_1(z+\frac{\pi}{2}) = \theta_2(z)$, $\theta_2(z+\frac{\pi}{2}) = -\theta_1(z)$, $\theta_3(z+\frac{\pi}{2}) = \theta_4(z)$, $\theta_4(z+\frac{\pi}{2}) = \theta_3(z)$.
- **Half-τ shifts:** with $\lambda(z) := q^{-1/4}e^{-iz}$,

$$\theta_1(z+\tfrac{\pi\tau}{2}) = i\lambda\,\theta_4(z), \quad \theta_4(z+\tfrac{\pi\tau}{2}) = i\lambda\,\theta_1(z), \quad \theta_2(z+\tfrac{\pi\tau}{2}) = \lambda\,\theta_3(z), \quad \theta_3(z+\tfrac{\pi\tau}{2}) = \lambda\,\theta_2(z).$$

(Each is a one-line reindexing; e.g. $\theta_4(z+\frac{\pi\tau}{2}) = \sum(-1)^n q^{n^2+n}e^{2inz} = q^{-1/4}\sum(-1)^n q^{(n+\frac{1}{2})^2}e^{2inz} = iq^{-1/4}e^{-iz}\theta_1(z)$.)

Also $\theta_1(\pi - z) = \theta_1(z)$ for **all complex** z (from $\sin((2n+1)(\pi - z)) = \sin((2n+1)z)$), and θ_4 is even with period π.

Key corollary (the special point). Let $w_0 := \dfrac{\pi\tau}{4}$. Applying the half-$\tau$ shifts at $z = -w_0$ (and evenness/oddness):

$$\theta_4(w_0) = i\,q^{-1/4}e^{iw_0}\,\theta_1(-w_0) = -i\,\theta_1(w_0), \qquad \theta_3(w_0) = q^{-1/4}e^{iw_0}\,\theta_2(-w_0) = \theta_2(w_0),$$

using $e^{iw_0} = q^{1/4}$. Hence

$$\theta_1(w_0)^2 = -\,\theta_4(w_0)^2, \qquad \theta_2(w_0) = \theta_3(w_0). \tag{0}$$

Also, at $z = 0$ the half-τ shifts give

$$\theta_1(2w_0) = \theta_1(\tfrac{\pi\tau}{2}) = i\,q^{-1/4}\,t_4. \tag{0$'$}$$

Step 1: reduction to an elliptic average

Substitute $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\theta \in [0, \pi/2]$. Then $x = \sin^2\theta$, $1 - x = \cos^2\theta$, $2 - x = 2(1 - \frac{1}{2}\sin^2\theta)$, so

$$I = \sqrt{2}\int_0^{\pi/2} \frac{\ln\left(\sqrt{2} + \sin^2\theta\right)}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}\,d\theta. \tag{1}$$

Now let $k = \frac{1}{\sqrt{2}}$ and define, for $\theta \in [0, \pi/2]$, $u(\theta) = \int_0^\theta \frac{dt}{\sqrt{1-k^2\sin^2 t}}$, a smooth increasing bijection $[0, \frac{\pi}{2}] \to [0, K]$ with $K = K(k)$. Its inverse is the Jacobi amplitude, $\theta = \operatorname{am}(u, k)$, and $\operatorname{sn}(u, k) := \sin\theta$ (this is simply the definition of sn on the real interval $[0, K]$ by inversion of the elliptic integral). Then $du = \dfrac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$ and (1) becomes

$$I = \sqrt{2}\,J, \qquad J := \int_0^K \ln\left(\sqrt{2} + \operatorname{sn}^2(u, k)\right)du. \tag{2}$$

Step 2: the theta parametrization at nome $q = e^{-\pi}$

From here on fix

$$q := e^{-\pi} \qquad (\text{so } \tau = i,\ w_0 = \tfrac{i\pi}{4},\ \lessdot\partial\approx\approx\sqsupset = \pi\mathbb{Z} + i\pi\mathbb{Z}).$$

(2a) $k(q) = k'(q) = \frac{1}{\sqrt{2}}$ **at** $q = e^{-\pi}$**.** Define $k(q) := t_2^2/t_3^2$, $k'(q) := t_4^2/t_3^2$. Poisson summation applied to the Gaussian $f(t) = e^{-\pi(t+1/2)^2}$ (Fourier transform $\hat{f}(\xi) = e^{i\pi\xi}e^{-\pi\xi^2}$) gives

$$t_2(e^{-\pi}) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+\frac{1}{2})^2} = \sum_{m\in\mathbb{Z}} (-1)^m e^{-\pi m^2} = t_4(e^{-\pi}),$$

so $k = k'$ at $q = e^{-\pi}$. Next, the **Jacobi quartic identity** $t_3^4 = t_2^4 + t_4^4$ holds for every q; proof by Liouville's theorem: the function $G(z) := \dfrac{t_2^2\,\theta_2^2(z) + t_4^2\,\theta_4^2(z)}{\theta_3^2(z)}$ is elliptic with periods $\pi, \pi\tau$ (all θ_j^2 pick up the same factor μ^2 under $z \mapsto z + \pi\tau$ and are π-periodic up to sign squared), and its only possible poles are double poles at the simple zeros $z_0 = \frac{\pi}{2} + \frac{\pi\tau}{2}$ of θ_3. By the shift identities, $\theta_2(z_0) = -\theta_1(\frac{\pi\tau}{2}) = -iq^{-1/4}t_4$ and $\theta_4(z_0) = \theta_3(\frac{\pi\tau}{2}) = q^{-1/4}t_2$, so the numerator at z_0 equals $t_2^2(-q^{-1/2}t_4^2) + t_4^2(q^{-1/2}t_2^2) = 0$: the pole order drops to at most 1. A nonconstant elliptic function has order ≥ 2, so G is constant. Evaluating at $z = \frac{\pi}{2}$: $G = \dfrac{0 + t_4^2t_3^2}{t_4^2} = t_3^2$; evaluating at $z = 0$: $G = \dfrac{t_2^4 + t_4^4}{t_3^2}$. Hence $t_3^4 = t_2^4 + t_4^4$, i.e. $k^2 + k'^2 = 1$. Combined with $k = k' > 0$ at $q = e^{-\pi}$:

$$k(e^{-\pi}) = k'(e^{-\pi}) = \tfrac{1}{\sqrt{2}}.$$

(2b) sn **as a theta quotient.** Define, at our q,

$$\phi(u) := \frac{t_3}{t_2}\frac{\theta_1(z)}{\theta_4(z)}, \qquad z = \frac{\pi u}{2K_\theta}, \qquad K_\theta := \frac{\pi}{2}\,t_3^2.$$

We prove $\phi(u) = \operatorname{sn}(u, k)$ for $u \in [0, K]$ and $K_\theta = K(k)$. We need three identities, each proved by the same Liouville scheme; let $X(z) := \theta_1^2(z)/\theta_4^2(z)$, an elliptic function (periods $\pi, \pi\tau$) whose only pole mod the lattice is the double pole at $\frac{\pi\tau}{2}$, with nonzero leading coefficient since $\theta_1(\frac{\pi\tau}{2}) = iq^{-1/4}t_4 \neq 0$. Any elliptic function F (periods $\pi, \pi\tau$) whose only singularity mod the lattice is a pole of order ≤ 2 at $\frac{\pi\tau}{2}$ can be written $F = A + BX$: choose B to cancel the double part; the remainder has at most one simple pole per period, hence (residue theorem: residues sum to 0) no pole at all, hence is constant.

Three-term identities. $F_2 := \theta_2^2/\theta_4^2$ and $F_3 := \theta_3^2/\theta_4^2$ are of this type. Using values at $z = 0$, $z = \frac{\pi}{2}$ (where $\theta_1 = t_2, \theta_2 = 0, \theta_3 = t_4, \theta_4 = t_3$) and $z = \frac{\pi}{2} + \frac{\pi\tau}{2}$ (where $\theta_3 = 0$, $\theta_1 = \theta_2(\frac{\pi\tau}{2}) = q^{-1/4}t_3$, $\theta_4 = \theta_3(\frac{\pi\tau}{2}) = q^{-1/4}t_2$):

$$t_4^2\,\theta_2^2(z) = t_2^2\,\theta_4^2(z) - t_3^2\,\theta_1^2(z), \qquad t_4^2\,\theta_3^2(z) = t_3^2\,\theta_4^2(z) - t_2^2\,\theta_1^2(z). \tag{3}$$

Wronskian identity. $W(z) := \theta_1'(z)\theta_4(z) - \theta_1(z)\theta_4'(z)$ satisfies (differentiate the quasi-periodicity relations) $W(z+\pi) = -W(z)$ and $W(z+\pi\tau) = \mu^2 W(z)$ — the same multipliers as $\theta_2(z)\theta_3(z)$. Hence $W/(\theta_2\theta_3)$ is elliptic, with possible simple poles only at the simple zeros $\frac{\pi}{2}$, $\frac{\pi}{2} + \frac{\pi\tau}{2}$ of $\theta_2\theta_3$. But $W(\frac{\pi}{2}) = \theta_2'(0)\theta_3(0) - \theta_2(0)\theta_3'(0) = 0$ (both θ_2, θ_3 even), and $W(z + \frac{\pi\tau}{2}) = q^{-1/2}e^{-2iz}\,W(z)$ (direct computation from the half-τ shifts, the extra $-i$ terms cancel), so $W(\frac{\pi}{2} + \frac{\pi\tau}{2}) = 0$ as well. Thus $W/(\theta_2\theta_3)$ is entire elliptic, i.e. constant $= W(0)/(t_2t_3) = \theta_1'(0)\,t_4/(t_2t_3)$. The **Jacobi derivative formula** $\theta_1'(0) = t_2t_3t_4$ follows from the triple products together with Euler's telescoping $\prod_{n\geq1}(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1}) = \prod_{n\geq1}(1+q^{2n})(1-q^{4n-2}) = \prod_{n\geq1}\frac{(1-q^{4n})(1-q^{4n-2})}{1-q^{2n}} = 1$, since $\theta_1'(0) = 2q^{1/4}P^3$ and $t_2t_3t_4 = 2q^{1/4}P^3\left[\prod(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1})\right]^2$. Hence

$$\theta_1'(z)\theta_4(z) - \theta_1(z)\theta_4'(z) = t_4^2\,\theta_2(z)\,\theta_3(z). \tag{4}$$

Now compute, with $' = d/du$ and using (4), (3):

$$\phi'(u) = \frac{t_3}{t_2}\cdot\frac{\pi}{2K_\theta}\cdot\frac{t_4^2\theta_2\theta_3}{\theta_4^2} = \frac{t_4^2}{t_2t_3}\cdot\frac{\theta_2(z)\theta_3(z)}{\theta_4^2(z)}, \qquad 1-\phi^2 = \frac{t_4^2\theta_2^2}{t_2^2\theta_4^2}, \qquad 1-k^2\phi^2 = \frac{t_4^2\theta_3^2}{t_3^2\theta_4^2},$$

whence $\phi'^2 = (1-\phi^2)(1-k^2\phi^2)$, $\phi(0) = 0$, $\phi'(0) = \theta_1'(0)/(t_2t_3t_4) = 1$. For $z \in (0, \frac{\pi}{2})$ all four thetas are positive, so $\phi' > 0$ and ϕ increases from 0 to $\phi(K_\theta) = \frac{t_3}{t_2}\cdot\frac{\theta_1(\pi/2)}{\theta_4(\pi/2)} = \frac{t_3}{t_2}\cdot\frac{t_2}{t_3} = 1$. Therefore, for $u \in [0, K_\theta]$, $u = \int_0^{\phi(u)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, i.e. $\phi = \mathrm{sn}(\cdot, k)$ and $K_\theta = K(k)$. In particular, with $z = \pi u/(2K)$, $q = e^{-\pi}$:

$$k\ \mathrm{sn}^2(u,k) = \frac{\theta_1^2(z)}{\theta_4^2(z)}, \qquad u \in [0,K], \qquad K = K(\tfrac{1}{\sqrt{2}}) = \frac{\pi}{2}t_3^2(e^{-\pi}). \tag{5}$$

Step 3: factorization of $\sqrt{2} + \mathrm{sn}^2$

Addition lemma. For all $z, w \in \mathbb{C}$:

$$t_4^2\,\theta_1(z+w)\,\theta_1(z-w) \;=\; \theta_1^2(z)\,\theta_4^2(w) \;-\; \theta_1^2(w)\,\theta_4^2(z). \tag{6}$$

Proof. Fix w generic and set $F(z) = \dfrac{\theta_1(z+w)\theta_1(z-w)}{\theta_4^2(z)}$. Under $z \mapsto z+\pi$ the numerator picks $(-1)^2$, the denominator 1; under $z \mapsto z+\pi\tau$ the numerator picks $(-q^{-1}e^{-2i(z+w)})(-q^{-1}e^{-2i(z-w)}) = \mu^2$, the denominator $(-\mu)^2 = \mu^2$. So F is elliptic with only possible singularity a double pole at $\frac{\pi\tau}{2}$ mod lattice; hence $F = A + BX$ as in Step 2b. At $z = 0$: $A = \theta_1(w)\theta_1(-w)/t_4^2 = -\theta_1^2(w)/t_4^2$. At $z = w$: $F(w) = 0$, so $B = -A/X(w) = \theta_4^2(w)/t_4^2$ (for w with $\theta_1(w)\theta_4(w) \neq 0$; the resulting polynomial identity extends to all w by continuity). Rearranging gives (6). □

Divide (6) by $\theta_4^2(z)\theta_4^2(w)$ and specialize $w = w_0 = \frac{i\pi}{4}$, using (0): $\theta_1^2(w_0)/\theta_4^2(w_0) = -1$:

$$\frac{t_4^2\,\theta_1(z+w_0)\,\theta_1(z-w_0)}{\theta_4^2(w_0)\,\theta_4^2(z)} = \frac{\theta_1^2(z)}{\theta_4^2(z)} + 1 \overset{(5)}{=} k\left(\mathrm{sn}^2\,u + \tfrac{1}{k}\right) = k\left(\mathrm{sn}^2\,u + \sqrt{2}\right),$$

since $1/k = \sqrt{2}$. Therefore, for all $u \in [0, K]$, $z = \pi u/(2K)$:

$$\sqrt{2} + \mathrm{sn}^2(u,k) \;=\; C\,\frac{\theta_1\Big(z + \frac{i\pi}{4}\Big)\,\theta_1\Big(z - \frac{i\pi}{4}\Big)}{\theta_4^2(z)}, \qquad C := \frac{t_4^2}{k\,\theta_4^2(w_0)}. \tag{7}$$

For real z, $\theta_1(z - \frac{i\pi}{4}) = \overline{\theta_1(z + \frac{i\pi}{4})}$, so the quotient in (7) equals $\big|\theta_1(z + \frac{i\pi}{4})\big|^2/\theta_4^2(z) > 0$; moreover $\theta_1(z + \frac{i\pi}{4}) \neq 0$ for real z (zeros of θ_1 lie on $\pi\mathbb{Z} + i\pi\mathbb{Z}$, imaginary parts $\in \pi\mathbb{Z} \not\ni \frac{\pi}{4}$). Also $\theta_4(w_0) = \theta_4(\frac{i\pi}{4}) = 1 + 2\sum_{n\ge1}(-1)^n e^{-\pi n^2}\cosh\frac{\pi n}{2} > 0$ (alternating with rapidly decreasing terms: $2e^{-\pi}\cosh\frac{\pi}{2} \approx 0.217$), so $C > 0$ and taking ln of (7) is legitimate termwise:

$$\ln\Big(\sqrt{2} + \mathrm{sn}^2 u\Big) = \ln C + \ln\big|\theta_1(z + \tfrac{i\pi}{4})\big|^2 - 2\ln\theta_4(z). \tag{7$'$}$$

Step 4: the logarithmic mean values

Integrate (7′) over $u \in [0, K]$, i.e. $z \in [0, \frac{\pi}{2}]$ ($du = \frac{2K}{\pi}dz$):

$$J = \frac{2K}{\pi}\left[\frac{\pi}{2}\ln C + \int_0^{\pi/2}\ln\big|\theta_1(z + \tfrac{i\pi}{4})\big|^2 dz - 2\int_0^{\pi/2}\ln\theta_4(z)\,dz\right]. \tag{8}$$

Lemma (mean values). Let $0 < q < 1$, $P = \prod_{n\ge1}(1 - q^{2n})$, and $0 < c < -\frac{1}{2}\ln q$. Then

$$\frac{1}{\pi}\int_0^\pi \ln\big|\theta_1(z + ic)\big|\,dz = \frac{\ln q}{4} + c + \ln P, \qquad \frac{1}{\pi}\int_0^\pi \ln\theta_4(z)\,dz = \ln P. \tag{9}$$

Proof. The basic fact: for $|\lambda| < 1$, $\int_0^\pi \ln|1 - \lambda e^{2iz}|\,dz = -\Re\sum_{m\ge1}\frac{\lambda^m}{m}\int_0^\pi e^{2imz}dz = 0$ (the series $\ln(1-\zeta) = -\sum_{m\ge1}\zeta^m/m$ converges uniformly on $|\zeta| \le |\lambda|$, so termwise integration is justified). Now use the triple products. For $\theta_4(z)$ (z real): each factor $|1 - q^{2n-1}e^{\pm 2iz}|$ has $|\lambda| = q^{2n-1} < 1$, so only $\ln P$ survives. For $\theta_1(z + ic)$: $|\sin(z + ic)| = \frac{e^c}{2}|1 - e^{-2c}e^{2iz}|$, contributing mean $c - \ln 2$; the factors $|1 - q^{2n}e^{-2c}e^{2iz}|$ have $|\lambda| = q^{2n}e^{-2c} < 1$, and $|1 - q^{2n}e^{2c}e^{-2iz}|$ have $|\lambda| = q^{2n}e^{2c} < 1$ precisely because $c < -\ln q$ suffices for $n \ge 1$ (indeed $q^2e^{2c} < 1$); all contribute 0. Adding $\ln(2q^{1/4}) + \ln P$ from the prefactor gives the first formula. □

Here $q = e^{-\pi}$ and $c = \frac{\pi}{4} < \frac{\pi}{2} = -\frac{1}{2}\ln q$, so the Lemma applies. Two symmetry reductions convert (8) to full-period means: θ_4 is even and π-periodic, so $\int_0^{\pi/2}\ln\theta_4 = \frac{1}{2}\int_0^\pi \ln\theta_4$; and $\theta_1(\pi - \zeta) = \theta_1(\zeta)$ for all complex ζ gives, for real z, $|\theta_1((\pi - z) + ic)| = |\theta_1(\pi - (z - ic))| = |\theta_1(z - ic)| = |\theta_1(z + ic)|$, so

$$\int_0^{\pi/2}\ln\big|\theta_1(z + ic)\big|^2 dz = \int_0^\pi \ln\big|\theta_1(z + ic)\big|\,dz.$$

Hence (8) becomes

$$J = \frac{2K}{\pi}\left[\frac{\pi}{2}\ln C + \pi\Big(\frac{\ln q}{4} + \frac{\pi}{4} + \ln P\Big) - \pi\ln P\right] = K\ln C + 2K\Big(\frac{\ln q}{4} + \frac{\pi}{4}\Big).$$

At $q = e^{-\pi}$ the last bracket **vanishes**: $\frac{\ln q}{4} + \frac{\pi}{4} = 0$. Therefore

$$J = K\,\ln C, \qquad C = \frac{t_4^2}{k\,\theta_4^2\big(\frac{i\pi}{4}\big)}. \tag{10}$$

Step 5: closed form for $\theta_4\left(\frac{i\pi}{4}\right)$, i.e. for C

Duplication lemma. For all z:

$$t_2t_3t_4\,\theta_1(2z) = 2\,\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z). \tag{11}$$

Proof. Let $H(z) = \dfrac{2\theta_1\theta_2\theta_3\theta_4(z)}{\theta_1(2z)}$. Multiplier check: under $z \mapsto z+\pi$, numerator picks $(-1)(-1)(1)(1) = 1$ and $\theta_1(2z+2\pi) = \theta_1(2z)$; under $z \mapsto z+\pi\tau$, the numerator picks $(-\mu)(\mu)(\mu)(-\mu) = \mu^4 = q^{-4}e^{-8iz}$ and the denominator $\theta_1(2z+2\pi\tau) = q^{-4}e^{-4i(2z)}\theta_1(2z)$ (iterate quasi-periodicity twice): same factor. So H is elliptic. The zeros of $\theta_1(2z)$ are exactly $z \in \frac{\pi}{2}\mathbb{Z} + \frac{\pi\tau}{2}\mathbb{Z}$, i.e. the four half-period classes $0, \frac{\pi}{2}, \frac{\pi\tau}{2}, \frac{\pi}{2}+\frac{\pi\tau}{2}$ mod the lattice — which are precisely the (simple) zeros of $\theta_1, \theta_2, \theta_4, \theta_3$ respectively in the numerator. Hence H is entire elliptic, therefore constant; letting $z \to 0$, $H \to \dfrac{2\theta_1'(0)z\,t_2t_3t_4}{\theta_1'(0)\,2z} = t_2t_3t_4$. $\square$

Evaluate (11) at $z = w_0 = \frac{i\pi}{4}$ (recall $2w_0 = \frac{\pi\tau}{2}$ and (0), (0′)): write $\alpha := \theta_4(w_0) > 0$, $\beta := \theta_3(w_0) = \theta_2(w_0)$, $\theta_1(w_0) = i\alpha$. Then

$$t_2t_3t_4 \cdot i\,q^{-1/4}t_4 \;=\; 2\,(i\alpha)\,\beta\,\beta\,\alpha \qquad\Longrightarrow\qquad \alpha^2\beta^2 = \frac{q^{-1/4}}{2}\,t_2t_3t_4^2. \tag{12}$$

Next, the second identity of (3) at $z = w_0$, with $\theta_1^2(w_0) = -\alpha^2$:

$$t_4^2\,\beta^2 = t_3^2\,\alpha^2 + t_2^2\,\alpha^2 \qquad\Longrightarrow\qquad \frac{\beta^2}{\alpha^2} = \frac{t_2^2+t_3^2}{t_4^2}. \tag{13}$$

(Consistently, the first identity of (3) gives the same for $\theta_2(w_0)^2$, confirming $\theta_2(w_0)^2 = \theta_3(w_0)^2$.) Combining (12), (13):

$$\alpha^4 = \frac{q^{-1/4}\,t_2\,t_3\,t_4^4}{2\,(t_2^2+t_3^2)}.$$

Hence, by (10),

$$C^2 = \frac{t_4^4}{k^2\alpha^4} = \frac{2\,(t_2^2+t_3^2)\,q^{1/4}}{k^2\,t_2t_3} = \frac{2(1+k)\,q^{1/4}}{k^{5/2}},$$

where we used $t_2^2 = k\,t_3^2$ (definition of $k(q)$, Step 2a), so $t_2^2+t_3^2 = (1+k)t_3^2$ and $t_2t_3 = \sqrt{k}\,t_3^2$. With $k = 2^{-1/2}$, $q^{1/4} = e^{-\pi/4}$:

$$C^2 = \frac{2\left(1+2^{-1/2}\right)}{2^{-5/4}}\,e^{-\pi/4} = 2^{9/4}\cdot 2^{-1/2}\left(1+\sqrt{2}\right)e^{-\pi/4} = 2^{7/4}\,(1+\sqrt{2})\,e^{-\pi/4},$$

and since $C > 0$,

$$C = 2^{7/8}\,(1+\sqrt{2})^{1/2}\,e^{-\pi/8}. \tag{14}$$

Step 6: the lemniscatic value of K and the final assembly

$$K\left(\tfrac{1}{\sqrt{2}}\right) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\frac{1}{2}t^2)}} = \sqrt{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} \overset{t=\sqrt{1-u^2}}{=} \sqrt{2}\int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

With $u^2 = \sin\varphi$: $\int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\varphi\,d\varphi = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{4\Gamma(\frac{3}{4})}$, and $\Gamma(\frac{3}{4}) = \dfrac{\pi}{\Gamma(\frac{1}{4})\sin\frac{\pi}{4}} = \dfrac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$, so

$$K\left(\tfrac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\frac{\Gamma(\frac{1}{4})^2\sqrt{\pi}}{4\pi\sqrt{2}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}. \tag{15}$$

Combining (2), (10), (14), (15):

$$I = \sqrt{2}\,K \ln C = \sqrt{2}\,K \left[\frac{7}{8}\ln 2 + \frac{1}{2}\ln(1+\sqrt{2}) - \frac{\pi}{8}\right] = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}} \cdot \frac{7\ln 2 + 4\ln(1+\sqrt{2}) - \pi}{8},$$

$$\boxed{\int_0^1 \frac{\ln\left(x+\sqrt{2}\right)}{\sqrt{(2-x)(1-x)x}}\,dx = \frac{\Gamma\left(\frac{1}{4}\right)^2}{16\sqrt{2\pi}}\left(7\ln 2 + 4\ln(1+\sqrt{2}) - \pi\right).}$$

Remark. The mechanism: the measure $\dfrac{dx}{\sqrt{x(1-x)(2-x)}}$ lives on the CM elliptic curve $y^2 = x(1-x)(2-x)$ with equally-spaced branch points, whose lambda-invariant is $\frac{1}{2}$, i.e. $\tau = i$, nome $e^{-\pi}$. The argument $x+\sqrt{2}$ vanishes at $x = -\sqrt{2}$, which corresponds to the 8-division point $u = \frac{iK'}{2}$ (indeed (0) and (5) show $\mathrm{sn}^2(\frac{iK'}{2}) = -\frac{1}{k} = -\sqrt{2}$); this is why $\ln(x+\sqrt{2})$ — and not a generic $\ln(x+a)$ — integrates in closed form. The $-\pi$ inside the bracket is the residual $q^{1/8} = e^{-\pi/8}$ of theta quasi-periodicity at the 8-division point.

Numerical verification

All computations with mpmath at `mp.dps = 60` (script `verify.py` in the scratch directory).

- Direct quadrature of the original singular integral (tanh–sinh on $[0, \frac{1}{2}, 1]$): `I_raw = 1.71611436816290240644184900503299619...` (correct to ≈ 33 digits).
- Quadrature of the smooth form (1): `I = 1.716114368162902406441849005033005386647187089076105711550...`
- Closed form $\dfrac{\Gamma(1/4)^2}{16\sqrt{2\pi}}(7\ln 2 + 4\ln(1+\sqrt{2}) - \pi)$: `CF = 1.716114368162902406441849005033005386647187089076105711550...`
- Agreement: **all 60 working digits** (difference 0.0 at 60 dps; ≥ 59 significant digits).

Every intermediate step was verified independently to ~ 60 digits: $I = \sqrt{2}J$ (diff $\sim 10^{-61}$); the theta parametrization (5) at a random point (10^{-62}); Lemma (6), identities (3), duplication (11) at random complex points (10^{-61}); $\theta_2(w_0) = \theta_3(w_0)$ and $\theta_4(w_0) = -i\theta_1(w_0)$ (exact to precision); the α^4 formula (0.0); the factorization (7) at a real point (10^{-61}); both mean values (9) (10^{-61}); $J = K \ln C$ (0.0); $C = 2^{7/8}(1+\sqrt{2})^{1/2}e^{-\pi/8} = 1.92416382125458199877\ldots$ (10^{-61}); $K(1/\sqrt{2}) = \Gamma(1/4)^2/(4\sqrt{\pi})$ and $q = e^{-\pi}$ (10^{-61}).

Notes

- The derivation is self-contained modulo two genuinely classical inputs, both stated explicitly: the **Jacobi triple product** (used for positivity of thetas on the real axis, for the Fourier mean-value Lemma (9), and — via Euler's telescoping product, included above — for the derivative formula $\theta_1'(0) = t_2t_3t_4$), and **Poisson summation for Gaussians** (one line, used only to see $t_2 = t_4$ at $q = e^{-\pi}$). Everything else (quasi-periodicity, shift identities, the quartic identity, the three-term and addition and duplication identities, the sn parametrization, the two mean values, the special values at the eighth-period point $w_0 = \pi\tau/4$) is proved above from scratch by series manipulation and Liouville-type arguments, and each was numerically confirmed to $\sim$60 digits.

- Interchange of summation and integration in the mean-value Lemma is justified by uniform convergence of $\ln(1-\zeta)$ on closed subdisks of $|\zeta| < 1$; the hypothesis $c < -\frac{1}{2}\ln q$ (here $\frac{\pi}{4} < \frac{\pi}{2}$) keeps every product factor in that regime.

- Branch/sign choices are pinned by explicit positivity: $\theta_4(\frac{i\pi}{4}) > 0$, $C > 0$, and the factored integrand in (7) is a positive real function on the integration segment, so all logarithms are real principal values.

- No step relies on recalled final answers; the closed form emerged from (10) and (14) and was then confirmed numerically to ≥ 59 significant digits.