Cleo Bench Problem 21

Integral $\int_0^{\pi/2} \arctan^2\left(\frac{6\sin x}{3+\cos 2x}\right) \mathrm{d}x$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^{\pi/2} \arctan^2\left(\frac{6\sin x}{3+\cos 2x}\right) \mathrm{d}x.$$

Result

Let χ_2 denote the Legendre chi function,

$$\chi_2(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^2} = \tfrac{1}{2}\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\right],$$

and put

$$r = \sqrt{2} - 1, \qquad q = \sqrt{5} - 2.$$

Then

$$\boxed{I = \pi\Big[\chi_2(r^2) + 2\,\chi_2(rq) + \chi_2(q^2)\Big] = \pi\Big[\chi_2(3-2\sqrt{2}) + 2\,\chi_2((\sqrt{2}-1)(\sqrt{5}-2)) + \chi_2(9-4\sqrt{5})\Big]}$$

equivalently, as a single series,

$$I = \pi \sum_{n \text{ odd}} \frac{\left[(\sqrt{2}-1)^n + (\sqrt{5}-2)^n\right]^2}{n^2}.$$

Numerically

$$I = 1.3309703968425666661253270759568320329950328192012082754302\,1\ldots$$

The derivation below also yields the general two–parameter formula

$$\int_0^{\pi/2} \arctan(a\sin x)\arctan(b\sin x)\,dx = \pi\,\chi_2\!\left(\frac{\sqrt{1+a^2}-1}{a} \cdot \frac{\sqrt{1+b^2}-1}{b}\right) \qquad (a, b > 0).$$

*Problem originally posed on Mathematics Stack Exchange (question 564816, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Derivation

Step 1. Algebraic simplification of the integrand

Since $\cos 2x = 1 - 2\sin^2 x$,

$$3 + \cos 2x = 4 - 2\sin^2 x = 2\,(2 - \sin^2 x), \qquad 2 - \sin^2 x \in [1, 2],$$

so the argument of the arctangent is

$$\frac{6\sin x}{3 + \cos 2x} = \frac{3\sin x}{2 - \sin^2 x} = \frac{\sin x + \frac{1}{2}\sin x}{1 - \sin x \cdot \frac{1}{2}\sin x}.$$

For real u, v with $uv < 1$ one has the addition formula

$$\arctan u + \arctan v = \arctan\frac{u + v}{1 - uv}.$$

(Indeed $A = \arctan u$, $B = \arctan v$ satisfy $A + B \in (-\pi, \pi)$ and $\cos(A + B) = \cos A \cos B\,(1 - uv) > 0$, hence $A + B \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\tan(A + B) = \frac{u+v}{1-uv}$.) Here $u = \sin x$, $v = \frac{1}{2}\sin x$ give $uv \le \frac{1}{2} < 1$, so for all x

$$\arctan\!\left(\frac{6\sin x}{3 + \cos 2x}\right) = \arctan(\sin x) + \arctan\!\left(\frac{\sin x}{2}\right),$$

and therefore

$$I = \int_0^{\pi/2} \left[\arctan(\sin x) + \arctan\!\left(\tfrac{1}{2}\sin x\right)\right]^2 dx.$$

Step 2. A Fourier expansion of $\arctan(a \sin x)$

Lemma 1. *Let $a > 0$ and set*

$$t = t_a = \frac{\sqrt{1 + a^2} - 1}{a} \in (0, 1), \qquad \textit{so that} \qquad \frac{2t}{1 - t^2} = a.$$

Then for all real x

$$\arctan(a \sin x) = 2 \sum_{n \text{ odd}} \frac{t^n}{n} \sin nx,$$

the series converging absolutely and uniformly on $\mathbb{R}$.

Proof of the parameter identity. $t^2 = \dfrac{(\sqrt{1 + a^2} - 1)^2}{a^2} = \dfrac{2 + a^2 - 2\sqrt{1 + a^2}}{a^2}$, hence $1 - t^2 = \dfrac{2(\sqrt{1 + a^2} - 1)}{a^2} = \dfrac{2t}{a}$, i.e. $\dfrac{2t}{1 - t^2} = a$. □

Proof of the expansion. With $0 < t < 1$ consider the product of the two complex numbers

$$P = (1 - te^{-ix})(1 + te^{ix}) = 1 + t\,(e^{ix} - e^{-ix}) - t^2 = (1 - t^2) + 2it \sin x = (1 - t^2)(1 + ia \sin x).$$

Both factors have positive real part: $\Re(1 - te^{-ix}) = 1 - t\cos x \ge 1 - t > 0$ and $\Re(1 + te^{ix}) = 1 + t\cos x > 0$; hence their principal arguments lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and their sum lies in $(-\pi, \pi)$. Also $\Re P = 1 - t^2 > 0$, so the principal argument of P is $\arg P = \arctan(a \sin x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since the sum of the two arguments equals $\arg P$ modulo 2π, and both numbers lie in $(-\pi, \pi)$, they are equal:

$$\arctan(a \sin x) = \Im \operatorname{Log}(1 - te^{-ix}) + \Im \operatorname{Log}(1 + te^{ix}).$$

For $|u| < 1$ the principal logarithm has the absolutely convergent expansion $\mathrm{Log}(1-u) = -\sum_{n\ge1} u^n/n$; taking $u = te^{-ix}$ and $u = -te^{ix}$,

$$\Im\,\mathrm{Log}(1-te^{-ix}) = \sum_{n\ge1} \frac{t^n \sin nx}{n}, \qquad \Im\,\mathrm{Log}(1+te^{ix}) = \sum_{n\ge1} (-1)^{n-1} \frac{t^n \sin nx}{n}.$$

Adding, the even harmonics cancel and the odd ones double:

$$\arctan(a \sin x) = 2 \sum_{n \text{ odd}} \frac{t^n}{n} \sin nx.$$

Uniform convergence follows from the Weierstrass M-test ($\sum_n t^n/n < \infty$). $\square$

For our two parameters,

$$a = 1: \quad t_1 = \sqrt{2} - 1 = r, \qquad\qquad a = \tfrac{1}{2}: \quad t_{1/2} = \frac{\sqrt{5}/2 - 1}{1/2} = \sqrt{5} - 2 = q.$$

Hence, adding the two expansions, the integrand of Step 1 is the square of

$$f(x) := \arctan(\sin x) + \arctan\bigl(\tfrac{1}{2}\sin x\bigr) = \sum_{n \text{ odd}} c_n \sin nx, \qquad c_n = \frac{2\,(r^n + q^n)}{n}.$$

Step 3. Orthogonality of odd sines on $[0, \pi/2]$ and Parseval

For odd integers n, m:

$$\int_0^{\pi/2} \sin nx\, \sin mx\, dx = \frac{1}{2} \int_0^{\pi/2} \bigl[\cos(n-m)x - \cos(n+m)x\bigr] dx.$$

If $n \neq m$, then $n \mp m$ are even and nonzero, and $\int_0^{\pi/2} \cos(2jx)\, dx = \frac{\sin j\pi}{2j} = 0$ for $j \in \mathbb{Z} \setminus \{0\}$; if $n = m$, $\int_0^{\pi/2} \sin^2 nx\, dx = \frac{\pi}{4} - \frac{\sin n\pi}{4n} = \frac{\pi}{4}$. Thus

$$\int_0^{\pi/2} \sin nx\, \sin mx\, dx = \frac{\pi}{4}\, \delta_{nm} \qquad (n, m \text{ odd}).$$

Since $\sum_n |c_n| < \infty$, the double series $\sum_{n,m} c_n c_m \sin nx \sin mx = f(x)^2$ converges absolutely and uniformly, so it may be integrated term by term:

$$I = \int_0^{\pi/2} f(x)^2 dx = \sum_{n,m \text{ odd}} c_n c_m \cdot \frac{\pi}{4} \delta_{nm} = \frac{\pi}{4} \sum_{n \text{ odd}} c_n^2 = \pi \sum_{n \text{ odd}} \frac{(r^n + q^n)^2}{n^2}.$$

Step 4. Summation in terms of the Legendre chi function

Expanding $(r^n + q^n)^2 = r^{2n} + 2(rq)^n + q^{2n}$ and using $\chi_2(z) = \sum_{n \text{ odd}} z^n/n^2$ (absolutely convergent for $|z| \le 1$; here r^2, rq, $q^2 \in (0,1)$):

$$I = \pi\Bigl[\chi_2(r^2) + 2\chi_2(rq) + \chi_2(q^2)\Bigr] = \pi\Bigl[\chi_2(3 - 2\sqrt{2}) + 2\chi_2\bigl((\sqrt{2}-1)(\sqrt{5}-2)\bigr) + \chi_2(9 - 4\sqrt{5})\Bigr].$$

This proves the boxed result. (The same computation with general a, b proves the two-parameter formula stated in the Result: by the Lemma and orthogonality, $\int_0^{\pi/2} \arctan(a \sin x) \arctan(b \sin x) dx = \frac{\pi}{4} \sum_{n \text{ odd}} \frac{2t_a^n}{n} \cdot \frac{2t_b^n}{n} = \pi \chi_2(t_a t_b)$.)

Step 5. Equivalent forms

Using $\chi_2(z) = \frac{1}{2}[\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)]$:

$$I = \frac{\pi}{2}\Big[\mathrm{Li}_2(3-2\sqrt{2}) - \mathrm{Li}_2(2\sqrt{2}-3) + \mathrm{Li}_2(9-4\sqrt{5}) - \mathrm{Li}_2(4\sqrt{5}-9)\Big] + \pi\Big[\mathrm{Li}_2(rq) - \mathrm{Li}_2(-rq)\Big],$$

with $rq = (\sqrt{2}-1)(\sqrt{5}-2) = 2 + \sqrt{10} - 2\sqrt{2} - \sqrt{5}$.

One may also transport the three χ_2 values by **Landen's identity**: for $0 < y < 1$,

$$\chi_2\Big(\frac{1-y}{1+y}\Big) + \chi_2(y) = \frac{\pi^2}{8} - \frac{1}{2}\ln y\,\ln\frac{1-y}{1+y}. \qquad (\star)$$

Proof of $(\star)$: both sides vanish trivially in the derivative check — differentiate the left side using $\chi_2'(z) = \operatorname{artanh}(z)/z$ and $\operatorname{artanh}\frac{1-y}{1+y} = -\frac{1}{2}\ln y$:

$$\frac{d}{dy}\chi_2\Big(\frac{1-y}{1+y}\Big) = \frac{-\frac{1}{2}\ln y}{\frac{1-y}{1+y}}\cdot\frac{-2}{(1+y)^2} = \frac{\ln y}{1-y^2}, \qquad \frac{d}{dy}\chi_2(y) = \frac{\operatorname{artanh} y}{y},$$

while the right side differentiates to $-\frac{1}{2}\big[\frac{1}{y}\ln\frac{1-y}{1+y} - \frac{2\ln y}{1-y^2}\big] = \frac{\operatorname{artanh} y}{y} + \frac{\ln y}{1-y^2}$; the two agree, and at $y \to 1^-$ both sides tend to $\chi_2(1) = \pi^2/8$. $\square$

Taking $y = \frac{1}{\sqrt{2}}$ (whose Landen partner is $\frac{1-y}{1+y} = (\sqrt{2}-1)^2$), $y = \frac{2}{\sqrt{5}}$ (partner $(\sqrt{5}-2)^2$), and $y = w := \frac{1-rq}{1+rq}$ (partner rq), one gets the equivalent evaluation (verified to 90 digits)

$$I = \pi\Big[\frac{\pi^2}{2} - \frac{\ln 2\,\ln(1+\sqrt{2})}{2} + 3\ln 2\,\ln\varphi - \frac{3}{2}\ln 5\,\ln\varphi + (\ln(1+\sqrt{2}) + 3\ln\varphi)\ln w - \chi_2\Big(\frac{1}{\sqrt{2}}\Big) - \chi_2\Big(\frac{2}{\sqrt{5}}\Big) - 2\chi_2(w)\Big],$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. This form is recorded only to exhibit the structure; the boxed form is the compact one.

Remark (why these arguments are natural). $r = \tanh\big(\frac{1}{2}\operatorname{arcsinh} 1\big)$ and $q = \tanh\big(\frac{1}{2}\operatorname{arcsinh}\frac{1}{2}\big)$; moreover $r = e^{-\operatorname{arcsinh} 1} = (1+\sqrt{2})^{-1}$ and $q = \varphi^{-3} = e^{-3\operatorname{arcsinh}(1/2)}$. The *first-power* values are the classical Landen ones,

$$\chi_2(\sqrt{2}-1) = \frac{\pi^2}{16} - \frac{\ln^2(1+\sqrt{2})}{4}, \qquad \chi_2(\sqrt{5}-2) = \frac{\pi^2}{24} - \frac{3}{4}\ln^2\varphi,$$

(both follow from $(\star)$: $y = \sqrt{2}-1$ is a fixed point of $y \mapsto \frac{1-y}{1+y}$, and $y = \varphi^{-1}$ pairs with $\sqrt{5}-2 = \varphi^{-3}$ combined with $\chi_2(\varphi^{-1}) = \frac{\pi^2}{12} - \frac{3}{4}\ln^2\varphi$). However, it is the *squares* r^2, q^2 and the *product* rq that occur in I, and these do not reduce further (see Notes).

Numerical verification

All computations with mpmath at `mp.dps = 90` (scripts in `results/scratch/work/q564816-.../`):

- Direct quadrature of $\int_0^{\pi/2} \arctan^2\Big(\frac{6\sin x}{3+\cos 2x}\Big)dx$ (smooth integrand), Gauss–Legendre and tanh–sinh, with split point $\pi/4$:

  $I_{\mathrm{GL}} = I_{\mathrm{TS}} = 1.3309703968425666661253270759568320329950328192012082754302103616\ldots$

  The two methods agree to all 90 digits.

- Closed form $\pi\big[\chi_2(r^2) + 2\chi_2(rq) + \chi_2(q^2)\big]$ evaluated via `mp.polylog`:

$$1.3309703968425666661253270759568320329950328192012082754302103616\ldots$$

- Agreement: $|I_{\text{quad}} - I_{\text{closed}}| < 5 \cdot 10^{-91}$, i.e. **90 significant digits** match (far beyond the required 25).

- The Parseval series $\pi \sum_{n \text{ odd}} (r^n + q^n)^2/n^2$ agrees with both to the same precision, and the Fourier expansion of Step 2 was checked pointwise at several x to 80 digits.

- Landen's identity ($\star$), the two classical special values in the Remark, and the equivalent form in Step 5 were each verified numerically to ≥ 89 digits.

Notes

- The derivation is complete and rigorous: the only analytic steps are the principal-branch argument addition (justified by positivity of real parts), term-by-term integration of an absolutely and uniformly convergent series, and elementary orthogonality; no interchange is left unjustified.

- **On further reduction.** I looked hard for an "elementary" form (rational combination of π^2 and products of logarithms). High-precision PSLQ (320-digit tolerance, coefficients up to 10^5–10^6) found no relation between I/π and $\{\pi^2\} \cup \{\text{all products of } \ln 2, \ln 3, \ln 5, \ln(1+\sqrt{2}), \ln\varphi, \ln(3+\sqrt{10}), \ln(1+\sqrt{10})\}$ — this log set spans the logs of all $\{2,3,5\}$-smooth-norm numbers of $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ that could arise. Furthermore, enumerating the full integer-relation lattice of Rogers L-function values at 35 candidate arguments (the five-term-relation orbit of $r, q, rq, \varphi^{-1}, \ldots$) shows the *only* relation satisfied by the target combination is the trivial ($\star$)-rewriting displayed in Step 5. Known relations in the lattice (e.g. the ladders $L(r^4) = 5L(r^2) - 1$ and $4L(q) - L(q^2) = 1$, and $L(\varphi^{-1}) + L(\varphi^{-2}) = 1$) were recovered by the search, confirming its power. I therefore believe the boxed χ_2 form is the minimal closed form; the non-existence of an elementary form is (as always for dilogarithm constants) heuristic, not proven.

- The identity $\int_0^{\pi/2} \arctan(a\sin x)\arctan(b\sin x)\,dx = \pi\chi_2(t_a t_b)$ has the amusing sanity check $a, b \to \infty$: $t_a, t_b \to 1$ and both sides tend to $\frac{\pi^3}{8}$.