Cleo Bench Problem 1

Integral $\int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln\left(\frac{2\,x^2+2\,x+1}{2\,x^2-2\,x+1}\right) \mathrm{d}x$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in exact closed form

$$I = \int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\, \ln\left(\frac{2x^2+2x+1}{2x^2-2x+1}\right) \mathrm{d}x \;\approx\; 8.372211626601275661625747121\ldots$$

(The poster is also interested in the variants with only the numerator or only the denominator under the logarithm; these are treated in the Notes.)

Result

$$\boxed{I \;=\; 4\pi\ \operatorname{arccot}\sqrt{\varphi}} \qquad \text{where } \varphi = \frac{1+\sqrt{5}}{2} \text{ is the golden ratio.}$$

Equivalent forms:

$$I \;=\; 4\pi \arctan\frac{1}{\sqrt{\varphi}} \;=\; 4\pi \arctan\sqrt{\varphi-1} \;=\; 4\pi \arcsin\frac{\sqrt{5}-1}{2} \;=\; 2\pi \arccos\left(\sqrt{5}-2\right) \;=\; 4\pi\,\Re \arcsin(1+i).$$

Numerically,

$$I = 8.372211626601275661625747121098412638081728053882207413717088296758124 73\ldots$$

Derivation

Notation. Log and $\sqrt{\cdot}$ denote the principal branches (cut along $(-\infty, 0]$); thus $\Re\sqrt{w} > 0$ for $w \notin (-\infty, 0]$, and $\operatorname{Arg} \in (-\pi, \pi]$. Throughout, $\varphi = \frac{1+\sqrt{5}}{2}$, so that

$$\varphi^2 = \varphi + 1, \qquad \varphi^{-1} = \varphi - 1, \qquad \varphi - \varphi^{-1} = 1.$$

All complex-valued integrals of a real variable are understood componentwise (real and imaginary parts), so real-variable theorems (FTC, substitution, Fubini, parity) apply verbatim.

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*Problem originally posed on Mathematics Stack Exchange (question 562694, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Step 0: convergence and reality

For all real x,

$$2x^2 \pm 2x + 1 = (x \pm 1)^2 + x^2 > 0,$$

so the logarithm is a real logarithm of a positive quantity — no branch issues in the integrand. Behaviour at the three delicate points of $(-1, 1)$:

- $x \to 0$: $\ln(1 \pm 2x + 2x^2) = \pm 2x + O(x^3)$ (the x^2-terms cancel: $2x^2 - \frac{(\pm 2x)^2}{2} = 0$), so the log factor is $4x + O(x^3)$ and the integrand extends continuously with value 4 at $x = 0$ (removable singularity).
- $x \to 1^-$: the integrand is $O\big((1-x)^{-1/2}\big)$ — integrable.
- $x \to -1^+$: the integrand is $O\big((1+x)^{1/2}\big) \to 0$.

Hence I converges absolutely.

Step 1: a parameter representation of the logarithm

For real x, t put

$$P_\pm(t) = 1 \pm 2xt + 2x^2t^2 = (1 \pm xt)^2 + (xt)^2.$$

Each $P_\pm$ is a sum of two squares that cannot vanish simultaneously ($xt = \mp 1$ and $xt = 0$ are incompatible), so $P_\pm(t) > 0$ for **all** real t, and $t \mapsto \ln \frac{P_+(t)}{P_-(t)}$ is smooth on $\mathbb{R}$. Moreover

$$P_+P_- = (1 + 2x^2t^2)^2 - (2xt)^2 = 1 + 4x^4t^4.$$

Differentiate:

$$\frac{d}{dt} \ln \frac{P_+(t)}{P_-(t)} = \frac{2x + 4x^2t}{P_+} - \frac{-2x + 4x^2t}{P_-} = \frac{(2x + 4x^2t)P_- - (-2x + 4x^2t)P_+}{1 + 4x^4t^4}.$$

The numerator is

$$2x\,(P_- + P_+) + 4x^2t\,(P_- - P_+) = 2x(2 + 4x^2t^2) + 4x^2t(-4xt) = 4x\left(1 - 2x^2t^2\right).$$

Since $P_\pm(0) = 1$ and $P_\pm(1) = 2x^2 \pm 2x + 1$, the fundamental theorem of calculus gives, **for every real** x,

$$\ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} = \int_0^1 \frac{4x\,(1 - 2x^2t^2)}{1 + 4x^4t^4}\,dt. \tag{1}$$

(Where this comes from: $P_+(t) = |1 + (1+i)xt|^2$, *and* (1) *is the real form of* $\frac{d}{dt} \operatorname{Log} \frac{1+(1+i)xt}{1-(1+i)xt} = \frac{2(1+i)x}{1 - 2ix^2t^2}$, *whose doubled real part is exactly* $\frac{4x(1-2x^2t^2)}{1+4x^4t^4}$. *The identity* (1) *itself, however, is purely real and was verified above by direct differentiation, so no branch discussion is needed.)*

Step 2: insert and swap (Tonelli–Fubini)

On $(-1, 1)$ we have $\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}}$. For $x \neq 0$ the factor $\frac{1}{x}$ cancels the x in (1):

$$\frac{1}{x}\sqrt{\frac{1+x}{1-x}} \ln \frac{2x^2 + 2x + 1}{2x^2 - 2x + 1} = \int_0^1 \frac{1+x}{\sqrt{1-x^2}} \cdot \frac{4\,(1 - 2x^2t^2)}{1 + 4x^4t^4}\,dt.$$

For $(x,t) \in (-1,1) \times [0,1]$ we have $|1-2x^2t^2| \le 1$ and $1+4x^4t^4 \ge 1$, so the absolute value of the double integrand is at most $\frac{4(1+x)}{\sqrt{1-x^2}}$, and

$$\int_{-1}^{1} \frac{4(1+x)}{\sqrt{1-x^2}}\,dx = 4\pi < \infty.$$

By Tonelli the double integral converges absolutely, and Fubini yields

$$I = \int_0^1 A(t)\,dt, \qquad A(t) := \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}} \cdot \frac{4\,(1-2x^2t^2)}{1+4x^4t^4}\,dx. \tag{2}$$

Step 3: parity

Split $\frac{1+x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$. The factor $\frac{4(1-2x^2t^2)}{1+4x^4t^4}$ is even in x; the contribution of the odd part $\frac{x}{\sqrt{1-x^2}}$ is an absolutely integrable odd function on $(-1,1)$, hence integrates to 0. Therefore

$$A(t) = 8\int_0^1 \frac{1-2t^2x^2}{\sqrt{1-x^2}\,(1+4t^4x^4)}\,dx. \tag{3}$$

Step 4: two lemmas

Lemma 1. *For $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$,*

$$\int_0^\infty \frac{d\tau}{1+\lambda^2\tau^2} = \frac{\pi}{2\lambda}.$$

Proof. Write $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1 > 0$, and $\mu = \operatorname{Arg}\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Factor $1+\lambda^2\tau^2 = (1+i\lambda\tau)(1-i\lambda\tau)$. For $\tau > 0$,

$$\Im(1+i\lambda\tau) = \lambda_1\tau > 0, \qquad \Im(1-i\lambda\tau) = -\lambda_1\tau < 0,$$

and both factors equal 1 at $\tau = 0$; in particular both stay off $(-\infty, 0]$ for all $\tau \ge 0$ (so also $1+\lambda^2\tau^2 \ne 0$). Hence $\tau \mapsto \operatorname{Log}(1 \pm i\lambda\tau)$ are C^1 on $[0,\infty)$ with derivatives $\frac{\pm i\lambda}{1\pm i\lambda\tau}$, and with the partial fractions $\frac{1}{1+\lambda^2\tau^2} = \frac{1}{2}\Big(\frac{1}{1+i\lambda\tau} + \frac{1}{1-i\lambda\tau}\Big)$,

$$\int_0^R \frac{d\tau}{1+\lambda^2\tau^2} = \frac{1}{2i\lambda}\Big[\operatorname{Log}(1+i\lambda R) - \operatorname{Log}(1-i\lambda R)\Big].$$

As $R \to \infty$: $\dfrac{|1+i\lambda R|}{|1-i\lambda R|} \to 1$, so the difference of the real parts of the Logs tends to 0. For the arguments, $1+i\lambda R = i\lambda R(1+\frac{1}{i\lambda R})$ and $1-i\lambda R = -i\lambda R(1-\frac{1}{i\lambda R})$, where the parenthetical factors tend to 1; since $\arg(i\lambda) = \mu + \frac{\pi}{2} \in (0,\pi)$ and $\arg(-i\lambda) = \mu - \frac{\pi}{2} \in (-\pi, 0)$ are interior points of $(-\pi,\pi)$ where Arg is continuous,

$$\operatorname{Arg}(1+i\lambda R) \to \mu + \tfrac{\pi}{2}, \qquad \operatorname{Arg}(1-i\lambda R) \to \mu - \tfrac{\pi}{2}.$$

Hence the bracket tends to $i\big[(\mu+\frac{\pi}{2}) - (\mu - \frac{\pi}{2})\big] = i\pi$, and the integral equals $\frac{i\pi}{2i\lambda} = \frac{\pi}{2\lambda}$. ☐

Lemma 2. *For $t \in [0,1]$ and $\beta := 2it^2$,*

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\,(1-\beta x^2)} = \frac{\pi}{2\sqrt{1-\beta}} \qquad \textit{(principal root)}.$$

Proof. Since $\Re(1-\beta x^2)=1$, we have $|1-\beta x^2|\ge 1$: the integrand is well defined and absolutely integrable (the only singularity is the integrable $(1-x^2)^{-1/2}$ at $x=1$). Substitute $x=\sin\theta$ ($\theta\in[0,\frac{\pi}{2}]$), then $\tau=\tan\theta$ ($\tau\in[0,\infty)$), both monotone C^1 bijections applied to an absolutely convergent integral; using $\sin^2\theta=\frac{\tau^2}{1+\tau^2}$, $d\theta=\frac{d\tau}{1+\tau^2}$,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}(1-\beta x^2)}=\int_0^{\pi/2}\frac{d\theta}{1-\beta\sin^2\theta}=\int_0^\infty \frac{d\tau}{1+(1-\beta)\tau^2}.$$

Let $\lambda=\sqrt{1-\beta}$ (principal). Since $\Re(1-\beta)=1>0$, we get $|\operatorname{Arg}(1-\beta)|<\frac{\pi}{2}$, hence $|\operatorname{Arg}\lambda|<\frac{\pi}{4}$ and $\Re\lambda>0$, with $\lambda^2=1-\beta$ exactly. Lemma 1 finishes the proof. □

Step 5: evaluation of $A(t)$

For real s,

$$\frac{1+i}{1-2is}=\frac{(1+i)(1+2is)}{1+4s^2}=\frac{(1-2s)+i\,(1+2s)}{1+4s^2}\qquad\Longrightarrow\qquad \frac{1-2s}{1+4s^2}=\Re\,\frac{1+i}{1-2is}.$$

Apply this with $s=t^2x^2\ge 0$ inside (3) and pull $\Re$ out of the absolutely convergent integral (linearity; the factor $\frac{1}{\sqrt{1-x^2}}$ is real):

$$A(t)=8\,\Re\left[(1+i)\int_0^1\frac{dx}{\sqrt{1-x^2}\,(1-2it^2x^2)}\right]\overset{\text{Lemma 2}}{=}8\,\Re\frac{(1+i)\,\pi}{2\sqrt{1-2it^2}}=4\pi\,\Re\,\frac{1+i}{\sqrt{1-2it^2}}.\quad(4)$$

(Consistency check: $A(0)=4\pi$, matching $8\int_0^1\frac{dx}{\sqrt{1-x^2}}=4\pi$.)

Step 6: the final integration — explicit antiderivative with branch control

By (2) and (4),

$$I=4\pi\,\Re\int_0^1\frac{(1+i)\,dt}{\sqrt{1-2it^2}}.\qquad(5)$$

Define on $[0,1]$

$$q(t):=i(1+i)\,t+\sqrt{1-2it^2}\;=\;-t+it+\sqrt{1-2it^2},\qquad \Phi(t):=-i\operatorname{Log}q(t).$$

(i) The path $q(t)$ stays in the open right half-plane. For $w=u+iv\notin(-\infty,0]$ the principal root satisfies

$$\Re\sqrt{w}=|w|^{1/2}\cos\tfrac{\alpha}{2}=\sqrt{\tfrac{|w|+u}{2}},\qquad \alpha=\operatorname{Arg}w\in(-\pi,\pi),$$

(using $\cos\frac{\alpha}{2}>0$ and $2\cos^2\frac{\alpha}{2}=1+\cos\alpha$). With $w=1-2it^2$ (so $u=1$, $|w|=\sqrt{1+4t^4}$):

$$\Re\,q(t)=-t+\sqrt{\frac{\sqrt{1+4t^4}+1}{2}}\;\ge\;1-t>0\quad(0\le t<1),$$

and at $t=1$: $\Re\,q(1)=\sqrt{\frac{1+\sqrt5}{2}}-1=\sqrt{\varphi}-1>0$. So $\Re q(t)>0$ on all of $[0,1]$; in particular $q(t)\ne 0$ and $q(t)$ avoids the branch cut of Log.

(ii) $\Phi'=(1+i)(1-2it^2)^{-1/2}$. Since $\Re(1-2it^2)=1>0$, the map $t\mapsto\sqrt{1-2it^2}$ is C^1 with derivative $\frac{-2it}{\sqrt{1-2it^2}}$ (chain rule for the principal root off its cut). By (i), $\operatorname{Log}q(t)$ is C^1 with derivative q'/q. Now

$$q'(t)=i(1+i)-\frac{2it}{\sqrt{1-2it^2}}=\frac{i\big[(1+i)\sqrt{1-2it^2}-2t\big]}{\sqrt{1-2it^2}},$$

while, using $i(1+i)^2 = i\cdot 2i = -2$,

$$(1+i)\,q(t) = i(1+i)^2 t + (1+i)\sqrt{1-2it^2} = -2t + (1+i)\sqrt{1-2it^2}.$$

Comparing, $q'(t) = \dfrac{i\,(1+i)\,q(t)}{\sqrt{1-2it^2}}$, hence

$$\Phi'(t) = -i\,\frac{q'(t)}{q(t)} = \frac{1+i}{\sqrt{1-2it^2}}, \qquad t\in[0,1].$$

(iii) FTC. Φ is C^1 on $[0,1]$ and $\Phi(0) = -i\,\mathrm{Log}\,1 = 0$, so

$$\int_0^1 \frac{(1+i)\,dt}{\sqrt{1-2it^2}} = \Phi(1) = -i\,\mathrm{Log}\,\big(\sqrt{1-2i}+i-1\big).$$

Writing $\mathrm{Log}\,q = \ln|q| + i\,\mathrm{Arg}\,q$ gives $\Re\big[-i\,\mathrm{Log}\,q\big] = \mathrm{Arg}\,q$, so by (5)

$$I = 4\pi\ \mathrm{Arg}\,\big(\sqrt{1-2i}+i-1\big). \tag{6}$$

(Remark: $\Phi(1) = -i\,\mathrm{Log}\,\big(i(1+i)+\sqrt{1-(1+i)^2}\big) = \arcsin(1+i)$ *in the principal branch, so* (6) *states* $I = 4\pi\,\Re\arcsin(1+i)$*.)*

Step 7: algebraic evaluation

Compute $\sqrt{1-2i}$: seek $p-iq$ with $p>0$ and $(p-iq)^2 = p^2-q^2-2ipq = 1-2i$, i.e. $p^2-q^2=1$, $pq=1$. Then $q=1/p$ and $p^4-p^2-1=0$, so $p^2 = \frac{1+\sqrt5}{2} = \varphi$ (the positive root). Hence

$$\sqrt{1-2i} = \sqrt{\varphi} - \frac{i}{\sqrt{\varphi}},$$

which is indeed the principal root ($\Re = \sqrt{\varphi} > 0$); as a check, $\big(\sqrt{\varphi} - \frac{i}{\sqrt{\varphi}}\big)^2 = \varphi - \varphi^{-1} - 2i = 1-2i$ by $\varphi - \varphi^{-1} = 1$.

Therefore, using $1 - \frac{1}{\sqrt{\varphi}} = \frac{\sqrt{\varphi}-1}{\sqrt{\varphi}}$,

$$q(1) = (\sqrt{\varphi}-1) + i\Big(1 - \frac{1}{\sqrt{\varphi}}\Big) = (\sqrt{\varphi}-1)\Big(1 + \frac{i}{\sqrt{\varphi}}\Big).$$

Since $\sqrt{\varphi} > 1$, the scalar $\sqrt{\varphi}-1$ is a positive real, so

$$\mathrm{Arg}\,q(1) = \mathrm{Arg}\,\Big(1 + \frac{i}{\sqrt{\varphi}}\Big) = \arctan\frac{1}{\sqrt{\varphi}} = \mathrm{arccot}\,\sqrt{\varphi}.$$

By (6),

$$\boxed{I = 4\pi\ \mathrm{arccot}\,\sqrt{\varphi}\,, \qquad \varphi = \frac{1+\sqrt5}{2}.}$$

Equivalent forms. Let $\theta = \mathrm{arccot}\,\sqrt{\varphi} \in (0,\frac{\pi}{2})$. Then $\csc^2\theta = 1+\cot^2\theta = 1+\varphi = \varphi^2$, so $\sin\theta = \varphi^{-1} = \frac{\sqrt5-1}{2}$, giving

$$I = 4\pi\arcsin\frac{\sqrt5-1}{2}.$$

Also $\varphi^{-2} = 2-\varphi$ gives $\cos 2\theta = 1-2\varphi^{-2} = 2\varphi-3 = \sqrt5-2$, whence $I = 2\pi\arccos(\sqrt5-2)$; and $\varphi^{-1} = \varphi-1$ gives $\frac{1}{\sqrt{\varphi}} = \sqrt{\varphi-1}$, whence $I = 4\pi\arctan\sqrt{\varphi-1}$.

(Interpretation: the computation is a rigorous implementation of "differentiation under the integral sign": for the family $F(a) = \int_{-1}^{1}\frac{1}{x}\sqrt{\frac{1+x}{1-x}}\ln\frac{1+ax}{1-ax}\,dx$ *one finds* $F'(a) = \frac{2\pi}{\sqrt{1-a^2}}$*, i.e.* $F(a) = 2\pi\arcsin a$*; our parameter path* $a = (1+i)t$*,* $t\in 0\to 1$*, evaluates the continuation at* $a = 1+i$ *and its conjugate simultaneously, and Steps 1–6 supply exactly the branch bookkeeping that makes "*$I = 2\pi\big(\arcsin(1+i)+\arcsin(1-i)\big) = 4\pi\Re\arcsin(1+i)$*" legitimate.)*

Numerical verification

All computations with mpmath (scripts `verify.py`, `steps.py`, `bonus.py` in the work directory).

Direct evaluation of I (dps = 130).

- Raw form: tanh–sinh quadrature of the original integrand over $[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1]$ gives $8.372211626601275661625747121098412638081728053882207413717088296757\mathbf{8}\mathbf{8}\ldots$ — agrees with the closed form to **67 significant digits** (limited only by quadrature error at the $(1-x)^{-1/2}$ endpoint).

- Transformed form: by the parity argument of Step 3 applied to the original integrand (the log factor is odd under $x \mapsto -x$, and $\frac{1+x}{x\sqrt{1-x^2}} = \underbrace{\frac{1}{x\sqrt{1-x^2}}}_{\text{odd}} + \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{even}}$), $I = 2\int_0^1 \frac{1}{x\sqrt{1-x^2}} \ln \frac{2x^2+2x+1}{2x^2-2x+1} dx$; substituting $x = \sin\theta$,

  $$I = 2\int_0^{\pi/2} \ln\left(\frac{1+2\sin\theta+2\sin^2\theta}{1-2\sin\theta+2\sin^2\theta}\right) \frac{d\theta}{\sin\theta},$$

  whose quadrature agrees with the closed form to **at least 130 significant digits** (difference exactly 0 at working precision).

Closed form (dps = 130).

$4\pi \operatorname{arccot}\sqrt{\frac{1+\sqrt{5}}{2}} = 8.3722116266012756616257471210984126380817280538822074137170882967581247324454$

This also reproduces all 28 digits quoted in the original post, $8.372211626601275661625747121\ldots$

Step-by-step checks (dps = 60, script `steps.py`):

- identity (1) at $x = 0.3$, -0.77, 0.999, ± 1: quadrature vs. logarithm — exact to working precision; also verified symbolically with sympy ($\frac{d}{dt}\ln\frac{P_+}{P_-} - \frac{4x(1-2x^2t^2)}{1+4x^4t^4} \equiv 0$ and $P_+P_- = 1+4x^4t^4$);

- (3) vs. (4) at $t = 0.25$, 0.7, 1.0: agreement to $\sim 10^{-32}$;

- Lemma 2 at $t = 0.33$, 0.9: $\sim 10^{-33}$; Lemma 1 for three test λ's with $\Re\lambda > 0$: $\leq 10^{-61}$;

- Φ' vs. $(1+i)(1-2it^2)^{-1/2}$ at $t = 0.2, 0.6, 0.95$ (central differences): $\sim 10^{-37}$; $\min_{[0,1]} \Re q(t) = 0.2720196495\ldots = \sqrt{\varphi} - 1$ attained at $t = 1$;

- $\int_0^1 (1+i)(1-2it^2)^{-1/2} dt = \Phi(1) = \arcsin(1+i) = 0.66623943249251525510\ldots + 1.06127506190503565203\ldots i$ (three ways, identical);

- $\sqrt{1-2i} = \sqrt{\varphi} - i/\sqrt{\varphi}$ and $\operatorname{Arg} q(1) = \arctan(1/\sqrt{\varphi}) = \Re\arcsin(1+i) = \arcsin\frac{\sqrt{5}-1}{2}$: all equal to $0.66623943249251525510400489597779272066 75\ldots$

Conclusion: direct numerics and the closed form agree to ≥ 67 significant digits (raw form) and ≥ 130 digits (exactly transformed form) — far beyond coincidence.

Notes

The two sub-questions. With the same tools one gets closed forms for the numerator-only and denominator-only variants:

$$N := \int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\,\ln(2x^2{+}2x{+}1)\,dx = \pi\ln\frac{\varphi+\sqrt{\varphi}}{2}{+}2\pi\operatorname{arccot}\sqrt{\varphi} \approx 5.3426136609159487441027947 14,$$

$$D := \int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\,\ln(2x^2{-}2x{+}1)\,dx = \pi\ln\frac{\varphi+\sqrt{\varphi}}{2}{-}2\pi\operatorname{arccot}\sqrt{\varphi} \approx -3.0295979656853269175229524 07.$$

Sketch (each step of the same rigor level as above): $N - D = I$; for the sum, $(2x^2 + 2x + 1)(2x^2 - 2x + 1) = 1 + 4x^4$ is even, so by the parity argument of Step 3, $N + D = \int_{-1}^{1} \frac{\ln(1+4x^4)}{\sqrt{1-x^2}}dx = 2\int_0^{\pi/2} \ln(1 + 4\sin^4\theta)\,d\theta$. Writing $1 + 4\sin^4\theta = |1 - 2i\sin^2\theta|^2$, one needs $G(s) := \int_0^{\pi/2} \operatorname{Log}(1 - 2is\sin^2\theta)\,d\theta$ at $s = 1$ (then $N + D = 4\,\Re\, G(1)$). Since $\Re(1 - 2is\sin^2\theta) = 1$, the derivative ∂_s-integrand $\frac{-2i\sin^2\theta}{1-2is\sin^2\theta}$ is bounded by 2; dominated convergence permits differentiation under the integral, and for $s > 0$, by Lemma 2,

$$G'(s) = \frac{1}{s}\int_0^{\pi/2} \Big[1 - \frac{1}{1 - 2is\sin^2\theta}\Big]d\theta = \frac{1}{s}\Big[\frac{\pi}{2} - \frac{\pi}{2\sqrt{1 - 2is}}\Big].$$

The function $\widetilde{G}(s) := \pi\operatorname{Log}\frac{1+\sqrt{1-2is}}{2}$ is well defined ($\Re\sqrt{1 - 2is} \geq 1$) with the same derivative: putting $w = 1-2is$, $s = \frac{i(w-1)}{2} = \frac{i}{2}(\sqrt{w}-1)(\sqrt{w}+1)$, one checks $\frac{\pi}{2s}\frac{\sqrt{w}-1}{\sqrt{w}} = \frac{-i\pi}{\sqrt{w}(1+\sqrt{w})} = \widetilde{G}'(s)$. Both vanish at $s = 0$, so $G \equiv \widetilde{G}$ on $[0, 1]$ and

$$N + D = 4\pi\ln\Big|\frac{1 + \sqrt{1 - 2i}}{2}\Big| = 2\pi\ln\frac{(1 + \sqrt{\varphi})^2 + \varphi^{-1}}{4} = 2\pi\ln\frac{\varphi + \sqrt{\varphi}}{2},$$

using $(1 + \sqrt{\varphi})^2 + \varphi^{-1} = 1 + 2\sqrt{\varphi} + \varphi + (\varphi - 1) = 2(\varphi + \sqrt{\varphi})$. Both N and D were confirmed numerically to ~ 30 digits (`bonus.py`).

Caveats. None of substance for the main result: every interchange (Fubini in Step 2, $\Re \leftrightarrow \int$ in Step 5), every substitution, and every branch of $\operatorname{Log}/\sqrt{\cdot}$ used is justified in the text; the two lemmas are proved from scratch; no special-function identities are quoted beyond the principal-branch definitions themselves. The only steps taken from the standard toolbox are Tonelli–Fubini, the fundamental theorem of calculus, dominated convergence (Notes only), and continuity of Arg off the cut. The final answer also matches the numeric value quoted in the original post to all 28 posted digits.