Cleo Bench Problem 23

Integral $\int_0^\infty \frac{\operatorname{arccot}\left(\sqrt{x}-2\sqrt{x+1}\right)}{x+1}\mathrm{d}x$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^\infty \frac{\operatorname{arccot}\left(\sqrt{x} - 2\sqrt{x+1}\right)}{x+1}\,\mathrm{d}x.$$

Convention. For every $x \geq 0$ one has $\sqrt{x} - 2\sqrt{x+1} < 0$; in fact (see Step 0) $\sqrt{x} - 2\sqrt{x+1} \leq -\sqrt{3}$. If arccot were taken with range $(0, \pi)$, the integrand would tend to $\pi/(x+1)$ as $x \to \infty$ and the integral would diverge. Hence the problem must be (and is) read with the **odd convention**

$$\operatorname{arccot}(y) = \arctan\frac{1}{y} \qquad (y \neq 0),$$

i.e. the range $(-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ — the convention of Mathematica's `ArcCot` and mpmath's `acot`. With this convention the integrand is continuous, negative and $O(x^{-3/2})$, so I converges. Everything below uses this convention.

Result

$$\boxed{\int_0^\infty \frac{\operatorname{arccot}\left(\sqrt{x} - 2\sqrt{x+1}\right)}{x+1}\,dx = \frac{\pi}{2}\ln\frac{3}{4} - \operatorname{Ti}_2\left(\frac{4}{3}\right) = -\pi\ln\frac{4}{3} - \operatorname{Ti}_2\left(\frac{3}{4}\right)}$$

where $\operatorname{Ti}_2(z) = \int_0^z \frac{\arctan t}{t}\,dt = \sum_{k\geq 0} \frac{(-1)^k z^{2k+1}}{(2k+1)^2} = \Im\operatorname{Li}_2(iz)$ is Lewin's inverse tangent integral. Numerically

$$I = -1.61434999002764854137373679018430737323694935699855\ldots$$

Equivalent dilogarithmic form: $I = \frac{\pi}{2}\ln\frac{3}{4} - \Im\operatorname{Li}_2\left(\frac{4i}{3}\right)$.

Derivation

Step 0. Convergence and range of the integrand

Let $y(x) = \sqrt{x} - 2\sqrt{x+1}$. Then $-y(x) = 2\sqrt{x+1} - \sqrt{x}$ attains its minimum where $\frac{d}{dx}(2\sqrt{x+1} - \sqrt{x}) = \frac{1}{\sqrt{x+1}} - \frac{1}{2\sqrt{x}} = 0$, i.e. $4x = x+1$, $x = \frac{1}{3}$, with value $2\sqrt{4/3} - \sqrt{1/3} = \sqrt{3}$.

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[*] Problem originally posed on Mathematics Stack Exchange (question 557439, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Hence $y(x) \leq -\sqrt{3}$ for all $x \geq 0$, so

$$\operatorname{arccot}\,(y(x)) = \arctan \frac{1}{y(x)} \in \Big[-\frac{\pi}{6}, 0\Big)$$

is continuous on $[0, \infty)$. As $x \to \infty$, $y(x) = -\sqrt{x}\,(2\sqrt{1+1/x} - 1) \sim -\sqrt{x}$, so the integrand is $\sim -x^{-3/2}$ and the integral converges absolutely. (With the $(0, \pi)$-range convention the integrand would be $\pi/(x+1) + O(x^{-3/2})$ — divergent — which forces the convention stated above.)

Step 1. Trigonometric substitution

Put $x = \tan^2\theta$, $\theta \in (0, \frac{\pi}{2})$. Then $\sqrt{x+1} = \sec\theta$, $x + 1 = \sec^2\theta$, $dx = 2\tan\theta\sec^2\theta\, d\theta$, so $\dfrac{dx}{x+1} = 2\tan\theta\, d\theta$, and

$$y = \tan\theta - 2\sec\theta = \frac{\sin\theta - 2}{\cos\theta} < 0.$$

Therefore, with the odd convention,

$$\operatorname{arccot}(y) = \arctan\frac{\cos\theta}{\sin\theta - 2} = -\arctan\frac{\cos\theta}{2 - \sin\theta} =: \eta(\theta),$$

which is smooth on $[0, \frac{\pi}{2}]$, with $\eta(0) = -\arctan\frac{1}{2}$ and $\eta(\frac{\pi}{2}) = 0$. Hence

$$I = 2\int_0^{\pi/2} \eta(\theta)\, \tan\theta\, d\theta. \tag{1}$$

Step 2. Differentiate η and integrate by parts

By the quotient rule, with $u(\theta) = \dfrac{\cos\theta}{2 - \sin\theta}$:

$$u' = \frac{-\sin\theta\,(2 - \sin\theta) + \cos^2\theta}{(2 - \sin\theta)^2} = \frac{1 - 2\sin\theta}{(2 - \sin\theta)^2}, \qquad 1 + u^2 = \frac{(2 - \sin\theta)^2 + \cos^2\theta}{(2 - \sin\theta)^2} = \frac{5 - 4\sin\theta}{(2 - \sin\theta)^2},$$

so

$$\eta'(\theta) = -\frac{u'}{1 + u^2} = \frac{2\sin\theta - 1}{5 - 4\sin\theta}. \tag{2}$$

Integrate (1) by parts with $\int \tan\theta\, d\theta = -\ln\cos\theta$:

$$I = 2\Big[-\eta(\theta)\ln\cos\theta\Big]_0^{\pi/2} + 2\int_0^{\pi/2} \eta'(\theta)\ln\cos\theta\, d\theta.$$

Both boundary terms vanish: at $\theta = 0$ because $\ln\cos 0 = 0$, and at $\theta \to \frac{\pi}{2}$ because

$$|\eta(\theta)\ln\cos\theta| \leq \frac{\cos\theta}{2 - \sin\theta}\,|\ln\cos\theta| \leq \cos\theta\,|\ln\cos\theta| \longrightarrow 0$$

(using $|\arctan v| \leq |v|$). Hence, by (2),

$$I = 2\int_0^{\pi/2} \frac{(2\sin\theta - 1)\ln\cos\theta}{5 - 4\sin\theta}\, d\theta. \tag{3}$$

Step 3. Split off the elementary part

Since $2\sin\theta - 1 = -\frac{1}{2}(5 - 4\sin\theta) + \frac{3}{2}$,

$$I = -\int_0^{\pi/2} \ln\cos\theta \, d\theta + 3\int_0^{\pi/2} \frac{\ln\cos\theta}{5 - 4\sin\theta}\, d\theta = \frac{\pi}{2}\ln 2 + 3W, \qquad W := \int_0^{\pi/2} \frac{\ln\cos\theta}{5 - 4\sin\theta}\, d\theta, \tag{4}$$

where we used the classical Euler integral $\int_0^{\pi/2} \ln\cos\theta \, d\theta = -\frac{\pi}{2}\ln 2$. (Proof: $J := \int_0^{\pi/2} \ln\sin = \int_0^{\pi/2} \ln\cos$ by $\theta \mapsto \frac{\pi}{2} - \theta$; then $2J = \int_0^{\pi/2} \ln\frac{\sin 2\theta}{2}\, d\theta = \frac{1}{2}\int_0^{\pi} \ln\sin t \, dt - \frac{\pi}{2}\ln 2 = J - \frac{\pi}{2}\ln 2$.)

Step 4. Evaluate W: rationalize the kernel

Multiply numerator and denominator by $5 + 4\sin\theta$ and use $25 - 16\sin^2\theta = 9 + 16\cos^2\theta$:

$$\frac{1}{5 - 4\sin\theta} = \frac{5 + 4\sin\theta}{9 + 16\cos^2\theta} \qquad \Longrightarrow \qquad W = 5\,W_1 + 4\,W_2, \tag{5}$$

$$W_1 := \int_0^{\pi/2} \frac{\ln\cos\theta}{9 + 16\cos^2\theta}\, d\theta, \qquad W_2 := \int_0^{\pi/2} \frac{\sin\theta\,\ln\cos\theta}{9 + 16\cos^2\theta}\, d\theta.$$

Lemma A. For $p \geq 0$: $\displaystyle\int_0^{\infty} \frac{\ln(1 + p^2v^2)}{1 + v^2}\, dv = \pi\ln(1 + p)$.

Proof. Call it $J(p)$; $J(0) = 0$. For p in a compact subset of $(0, \infty)$ the differentiated integrand $\frac{2pv^2}{(1+p^2v^2)(1+v^2)}$ is dominated by $\frac{C}{1+v^2}\cdot$const uniformly, so differentiation under the integral sign is legitimate (dominated convergence), and by the partial fraction $\frac{v^2}{(1+p^2v^2)(1+v^2)} = \frac{1}{p^2-1}\big[\frac{1}{1+v^2} - \frac{1}{1+p^2v^2}\big]$ (for $p \neq 1$; $p = 1$ removable by continuity),

$$J'(p) = \frac{2p}{p^2 - 1}\Big[\frac{\pi}{2} - \frac{\pi}{2p}\Big] = \frac{\pi}{p + 1}.$$

Integrating from 0: $J(p) = \pi\ln(1 + p)$. ■

Evaluation of W_1. Write $9 + 16\cos^2\theta = 9\sin^2\theta + 25\cos^2\theta$ and substitute $v = \tan\theta$ ($\cos^2\theta = \frac{1}{1+v^2}$, $\ln\cos\theta = -\frac{1}{2}\ln(1 + v^2)$, $d\theta = \frac{dv}{1+v^2}$):

$$W_1 = \int_0^{\pi/2} \frac{\ln\cos\theta}{\cos^2\theta\,(9\tan^2\theta + 25)}\,\frac{d\theta\,\cos^2\theta}{1} = -\frac{1}{2}\int_0^{\infty} \frac{\ln(1 + v^2)}{9v^2 + 25}\, dv.$$

Now $v = \frac{5}{3}w$ gives $\int_0^{\infty} \frac{\ln(1+v^2)}{9v^2+25}dv = \frac{1}{15}\int_0^{\infty} \frac{\ln\left(1+\frac{25}{9}w^2\right)}{1+w^2}dw = \frac{\pi}{15}\ln\frac{8}{3}$ by Lemma A with $p = \frac{5}{3}$. Hence

$$W_1 = -\frac{\pi}{30}\ln\frac{8}{3}. \tag{6}$$

Lemma B. For $y > 0$: $\displaystyle\int_0^{y} \frac{\ln v}{1 + v^2}\, dv = \arctan y\,\ln y - \mathrm{Ti}_2(y)$.

Proof. Integrate by parts, $d(\arctan v) = \frac{dv}{1+v^2}$: the boundary term $\arctan v\,\ln v \to 0$ as $v \to 0^+$ (since $\arctan v \sim v$), and $\int_0^y \frac{\arctan v}{v}dv = \mathrm{Ti}_2(y)$ by definition. ■

Evaluation of W_2. Substitute $t = \cos\theta$:

$$W_2 = \int_0^1 \frac{\ln t}{9 + 16t^2}\, dt \overset{t=\frac{3}{4}v}{=} \frac{1}{12}\int_0^{4/3} \frac{\ln\frac{3}{4} + \ln v}{1 + v^2}\, dv = \frac{1}{12}\Big[\ln\tfrac{3}{4}\,\arctan\tfrac{4}{3} + \arctan\tfrac{4}{3}\,\ln\tfrac{4}{3} - \mathrm{Ti}_2\left(\tfrac{4}{3}\right)\Big],$$

by Lemma B with $y = \frac{4}{3}$. The two logarithmic terms cancel ($\ln\frac{3}{4} + \ln\frac{4}{3} = 0$), so

$$W_2 = -\frac{1}{12}\,\mathrm{Ti}_2\!\left(\frac{4}{3}\right). \tag{7}$$

Step 5. Assemble

From (5), (6), (7):

$$W = -\frac{\pi}{6}\ln\frac{8}{3} - \frac{1}{3}\operatorname{Ti}_2\!\left(\frac{4}{3}\right),$$

and from (4):

$$I = \frac{\pi}{2}\ln 2 - \frac{\pi}{2}\ln\frac{8}{3} - \operatorname{Ti}_2\!\left(\frac{4}{3}\right) = \frac{\pi}{2}\ln\frac{3}{4} - \operatorname{Ti}_2\!\left(\frac{4}{3}\right). \tag{8}$$

Step 6. Equivalent forms

Lemma C (inversion). For $y > 0$: $\operatorname{Ti}_2(y) - \operatorname{Ti}_2(1/y) = \frac{\pi}{2}\ln y$.

Proof. Both sides vanish at $y = 1$ and have equal derivatives: $\frac{d}{dy}\left[\operatorname{Ti}_2(y) - \operatorname{Ti}_2(1/y)\right] = \frac{\arctan y}{y} + \frac{\arctan(1/y)}{y} = \frac{\pi}{2y}$, using $\arctan y + \arctan\frac{1}{y} = \frac{\pi}{2}$ for $y > 0$. ■

With $y = \frac{4}{3}$, (8) becomes

$$I = \frac{\pi}{2}\ln\frac{3}{4} - \frac{\pi}{2}\ln\frac{4}{3} - \operatorname{Ti}_2\!\left(\frac{3}{4}\right) = -\pi\ln\frac{4}{3} - \operatorname{Ti}_2\!\left(\frac{3}{4}\right).$$

Also $\operatorname{Ti}_2(y) = \Im\operatorname{Li}_2(iy)$ for $y > 0$ (the Maclaurin identity $\operatorname{Li}_2(iy) = \frac{1}{4}\operatorname{Li}_2(-y^2) + i\operatorname{Ti}_2(y)$ for $y \le 1$ extends to all $y > 0$ by analytic continuation along the imaginary axis, which never meets the cut $[1, \infty)$ of Li_2), giving

$$I = \frac{\pi}{2}\ln\frac{3}{4} - \Im\operatorname{Li}_2\!\left(\frac{4i}{3}\right) = -\pi\ln\frac{4}{3} - \Im\operatorname{Li}_2\!\left(\frac{3i}{4}\right).$$

Remark on the shape of the answer. Since $\arctan\frac{4}{3} = 2\arctan\frac{1}{2}$ is not a rational multiple of π, the standard reduction $\operatorname{Ti}_2(\tan\phi) = \phi\ln\tan\phi + \frac{1}{2}\operatorname{Cl}_2(2\phi) + \frac{1}{2}\operatorname{Cl}_2(\pi - 2\phi)$ produces no further elementary simplification; $\operatorname{Ti}_2(\frac{4}{3})$ (equivalently $\operatorname{Ti}_2(\frac{3}{4})$) is the irreducible transcendental constant of this problem, exactly as Catalan's constant $G = \operatorname{Ti}_2(1)$ would be. See the Notes for a PSLQ search confirming no small-height relation with the usual constant basis.

Numerical verification

All computations with mpmath.

1. **Direct evaluation of the integral as posed** (odd-convention `acot`, i.e. `mp.acot`), at `mp.dps = 80`, splitting $[0, \infty)$ at $x = \frac{1}{3}, 1, 4, 20, 100$ and mapping the tail $x > 100$ by $x = 1/t^2$:

   $$I_{\text{quad}} = -1.6143499900276485413737367901843073732369493569985507490797150\ldots$$

2. **Independent direct evaluation** in the form (1) (pure change of variable, before any integration by parts), $2\int_0^{\pi/2} \eta\tan\theta\,d\theta$, at `mp.dps = 60`: agrees with the closed form to $\sim 6\times10^{-61}$ (all 60 digits).

3. **Closed form** $\frac{\pi}{2}\ln\frac{3}{4} - \operatorname{Ti}_2(\frac{4}{3})$ with $\operatorname{Ti}_2(\frac{4}{3}) = \int_0^{4/3}\frac{\arctan t}{t}dt$ by quadrature, and independently as $\Im\operatorname{Li}_2(4i/3)$ and $-\pi\ln\frac{4}{3} - \Im\operatorname{Li}_2(0.75i)$ via `mp.polylog`; all four forms agree among themselves to full working precision ($\le 3\times10^{-61}$ at 60 dps).

4. **Agreement:** $|I_{\text{quad}} - I_{\text{closed}}| \approx 2.1\times10^{-81}$ at 80 digits of working precision — **80 matching significant digits**.

5. Every intermediate identity was checked numerically at 60 digits: the derivative formula (2), equation (3), the value of W, W_1 vs $-\frac{\pi}{30}\ln\frac{8}{3}$, W_2 vs $-\frac{1}{12}\operatorname{Ti}_2(\frac{4}{3})$, Lemma A at $p=\frac{5}{3}$, and Lemma C at $y=\frac{4}{3}$. All residuals were $< 10^{-58}$.

Reference values (60 significant digits):

$$I = -1.61434999002764854137373679018430737323694935699855074907972\ldots$$

$$\operatorname{Ti}_2\left(\frac{4}{3}\right) = 1.16246004733564774158995818385573624847446067214211087965611\ldots$$

Notes

- **Convention caveat.** The result assumes $\operatorname{arccot} y = \arctan\frac{1}{y}$ (odd convention, range $(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$). This is forced: with the continuous $(0,\pi)$ convention the integral diverges to $+\infty$. This is the standard reading for such forum problems (Mathematica's `ArcCot`).
- **Irreducibility of** $\operatorname{Ti}_2(4/3)$**.** A PSLQ search at 60 digits for integer relations (coefficients up to 10^6–10^8) between $\operatorname{Ti}_2(\frac{3}{4})$ and the basis $\{G,\ \pi\ln 2,\ \pi\ln 3,\ \pi\ln 5,\ \arctan\frac{1}{2}\ln p,\ \arctan\frac{1}{3}\ln p\ (p=2,3,5),\ \pi^2\}$ found only the trivial relation coming from $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$. So the answer very likely cannot be reduced to G, π, and logarithms; $\operatorname{Ti}_2(\frac{4}{3})$ is the natural final constant. (Known closed-form points of Ti_2 are $\tan\phi$ with ϕ a special rational multiple of π: $1,\ 2\pm\sqrt{3}$, etc.; $\frac{4}{3}$ is not of this type.)
- **A curious empirical relation (not used above).** PSLQ does find, and I verified to 150 digits,
$$\operatorname{Ti}_2\left(\frac{3}{4}\right)+4\operatorname{Ti}_2\left(\frac{1}{2}\right)+2\operatorname{Ti}_2\left(\frac{1}{3}\right)=6G-\pi\ln 2,$$
a Lewin-type multi-argument relation (consistent with $2\arctan\frac{1}{3}=\arctan\frac{3}{4}$ and $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$). I did not prove it and the main result does not depend on it; it shows the answer can also be written as $I=-\pi\ln\frac{4}{3}-6G+4\operatorname{Ti}_2(\frac{1}{2})+2\operatorname{Ti}_2(\frac{1}{3})+\pi\ln 2$, which is no simpler.
- **Rigor status.** The derivation is complete and elementary: one change of variables, one integration by parts with vanishing boundary terms (justified), a rational kernel split, Lemma A (differentiation under the integral, justified by dominated convergence), Lemma B (integration by parts), Lemma C (inversion). No steps are heuristic. Combined with 80-digit numerical agreement, I consider the result established.