Cleo Bench Problem 7

Calculating $\int_{\pi/2}^{\pi} \frac{x \sin x}{5-4\cos x}\,\mathrm{d}x$

Derivation by Claude (Fable 5), closed-book*

July 2026

Problem

Evaluate in exact closed form

$$I \;=\; \int_{\pi/2}^{\pi} \frac{x\sin x}{5-4\cos x}\,\mathrm{d}x.$$

(The classical companion $\int_0^{\pi} \frac{x\sin x}{5-4\cos x}\,dx = \frac{\pi}{2}\ln\frac{3}{2}$ is easy; the point of the problem is the half interval $[\pi/2, \pi]$.)

Result

$$\boxed{\int_{\pi/2}^{\pi} \frac{x\sin x}{5-4\cos x}\,dx \;=\; \frac{\pi}{8}\ln\frac{81}{20} \;-\; \frac{1}{2}\,\mathrm{Ti}_2\Big(\frac{1}{2}\Big)}$$

where Ti_2 is the inverse tangent integral,

$$\mathrm{Ti}_2(y) = \int_0^y \frac{\arctan t}{t}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)^2} = \Im\,\mathrm{Li}_2(iy),$$

$$\mathrm{Ti}_2\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2\,2^{2k+1}} = 0.48722235829452235711\ldots$$

Equivalent forms of the answer:

$$I = \frac{\pi}{2}\ln 3 - \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 5 - \frac{1}{2}\,\Im\,\mathrm{Li}_2\Big(\frac{i}{2}\Big) = \frac{\pi}{8}\ln\frac{81}{20} + \frac{i}{4}\Big[\,\mathrm{Li}_2\left(\frac{i}{2}\right) - \mathrm{Li}_2\left(-\frac{i}{2}\right)\Big].$$

Numerically $I = 0.30566365562445678428795111199994820290395346693\ldots$

Derivation

Step 0. Structure of the denominator

For all real x,

$$(2-e^{ix})(2-e^{-ix}) = 4 - 2\big(e^{ix}+e^{-ix}\big) + 1 = 5 - 4\cos x,$$

*Problem originally posed on Mathematics Stack Exchange (question 418134, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

so

$$5-4\cos x = 4\left|1-\tfrac{1}{2}e^{ix}\right|^2 \ \in [1,9],$$

strictly positive. Hence the integrand is continuous on $[\pi/2,\pi]$ (no singularities) and all manipulations below take place with absolutely bounded, continuous functions.

Step 1. Integration by parts

Since $\dfrac{d}{dx}\ln(5-4\cos x) = \dfrac{4\sin x}{5-4\cos x}$, we have $\dfrac{\sin x}{5-4\cos x} = \dfrac{1}{4}\dfrac{d}{dx}\ln(5-4\cos x)$, and both factors x and $\ln(5-4\cos x)$ are C^1 on $[\pi/2,\pi]$, so integration by parts is legitimate:

$$I = \frac{1}{4}\Big[x\ln(5-4\cos x)\Big]_{\pi/2}^{\pi} - \frac{1}{4}\int_{\pi/2}^{\pi}\ln(5-4\cos x)\,dx.$$

With $5-4\cos\pi = 9$ and $5-4\cos\frac{\pi}{2} = 5$,

$$I = \frac{\pi\ln 9}{4} - \frac{\pi\ln 5}{8} - \frac{1}{4}L, \qquad L := \int_{\pi/2}^{\pi}\ln(5-4\cos x)\,dx. \tag{1}$$

Step 2. The log integral L

By Step 0,

$$\ln(5-4\cos x) = 2\ln 2 + 2\ln\left|1-\tfrac{1}{2}e^{ix}\right|. \tag{2}$$

For $|z|\le\frac{1}{2}$ the principal logarithm has the absolutely convergent expansion $\log(1-z) = -\sum_{n\ge1}\frac{z^n}{n}$; with $z=\frac{1}{2}e^{ix}$ the terms are bounded by $\frac{1}{n2^n}$, so by the Weierstrass M-test the series converges **uniformly in** $x\in\mathbb{R}$. Taking real parts,

$$\ln\left|1-\tfrac{1}{2}e^{ix}\right| = -\sum_{n=1}^{\infty}\frac{\cos nx}{n\,2^n},$$

uniformly on $[\pi/2,\pi]$. Uniform convergence on the compact interval justifies term-by-term integration:

$$\begin{aligned}\int_{\pi/2}^{\pi}\ln\left|1-\tfrac{1}{2}e^{ix}\right|dx &= -\sum_{n=1}^{\infty}\frac{1}{n2^n}\int_{\pi/2}^{\pi}\cos nx\,dx\\ &= -\sum_{n=1}^{\infty}\frac{\sin n\pi-\sin\frac{n\pi}{2}}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2 2^n}.\end{aligned}$$

Now $\sin\frac{n\pi}{2}=0$ for even n and $\sin\frac{(2k+1)\pi}{2} = (-1)^k$, so

$$\int_{\pi/2}^{\pi}\ln\left|1-\tfrac{1}{2}e^{ix}\right|dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2\,2^{2k+1}} = \mathrm{Ti}_2\left(\frac{1}{2}\right), \tag{3}$$

by the defining series of the inverse tangent integral. (That series equals $\int_0^{1/2}\frac{\arctan t}{t}dt$ by integrating $\frac{\arctan t}{t} = \sum_{k\ge0}\frac{(-1)^k t^{2k}}{2k+1}$ term by term — again uniform on $[0,\frac{1}{2}]$ — and equals $\Im\,\mathrm{Li}_2(i/2)$ because $\mathrm{Li}_2(iy) = \sum_n\frac{(iy)^n}{n^2}$ has imaginary part $\sum_n\frac{y^n\sin(n\pi/2)}{n^2}$.)

Series-free cross-check of (3). Li_2 is analytic on $|z|<1$ and $\frac{d}{dx}\mathrm{Li}_2(\frac{1}{2}e^{ix}) = -i\log(1-\frac{1}{2}e^{ix})$, so $\ln|1-\frac{1}{2}e^{ix}| = -\frac{d}{dx}\Im\,\mathrm{Li}_2(\frac{1}{2}e^{ix})$, whence

$$\int_{\pi/2}^{\pi}\ln\left|1-\tfrac{1}{2}e^{ix}\right|dx = \Im\,\mathrm{Li}_2\left(\frac{i}{2}\right) - \Im\,\mathrm{Li}_2\left(-\frac{1}{2}\right) = \mathrm{Ti}_2\left(\frac{1}{2}\right),$$

since $\mathrm{Li}_2(-\frac{1}{2})$ is real. Same result.

Combining (2) and (3):

$$L = 2\ln 2\cdot\frac{\pi}{2} + 2\,\mathrm{Ti}_2\left(\frac{1}{2}\right) = \pi\ln 2 + 2\,\mathrm{Ti}_2\left(\frac{1}{2}\right). \tag{4}$$

Step 3. Assemble

Insert (4) into (1):

$$I = \frac{\pi \ln 9}{4} - \frac{\pi \ln 5}{8} - \frac{\pi \ln 2}{4} - \frac{1}{2}\operatorname{Ti}_2\left(\frac{1}{2}\right) = \frac{\pi}{8}\ln\frac{9^2}{4\cdot 5} - \frac{1}{2}\operatorname{Ti}_2\left(\frac{1}{2}\right) = \frac{\pi}{8}\ln\frac{81}{20} - \frac{1}{2}\operatorname{Ti}_2\left(\frac{1}{2}\right).$$

■

Consistency checks with the full range

An independent route (used as a cross-check) is the uniformly convergent Poisson-type series obtained from $\sum_{n\ge1} r^n \sin nx = \frac{r\sin x}{1-2r\cos x+r^2}$ at $r=\frac{1}{2}$:

$$\frac{\sin x}{5-4\cos x} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin nx}{2^n},$$

and $\int_{\pi/2}^{\pi} x\sin nx\,dx = -\frac{\pi(-1)^n}{n} + \frac{\pi\cos(n\pi/2)}{2n} - \frac{\sin(n\pi/2)}{n^2}$. Summing with $\sum_n \frac{(-1)^n}{n2^n} = -\ln\frac{3}{2}$ and $\sum_n \frac{\cos(n\pi/2)}{n2^n} = -\Re\log(1-\frac{i}{2}) = -\frac{1}{2}\ln\frac{5}{4}$ reproduces exactly the same value. In particular one gets the classical

$$\int_0^{\pi}\frac{x\sin x}{5-4\cos x}\,dx = \frac{\pi}{2}\ln\frac{3}{2}, \qquad \text{hence} \qquad \int_0^{\pi/2}\frac{x\sin x}{5-4\cos x}\,dx = \frac{\pi}{8}\ln\frac{5}{4} + \frac{1}{2}\operatorname{Ti}_2\left(\frac{1}{2}\right):$$

the two half-range integrals carry the constant $\operatorname{Ti}_2(1/2)$ with opposite signs, and it cancels over the full range — which is why $[0,\pi]$ is elementary while $[\pi/2,\pi]$ is not.

Why $\operatorname{Ti}_2(1/2)$ is the right "closed form"

$\operatorname{Ti}_2(1/2) = \Im\operatorname{Li}_2(i/2)$ is a standard dilogarithmic constant. It satisfies $\operatorname{Ti}_2(2) - \operatorname{Ti}_2(1/2) = \frac{\pi}{2}\ln 2$ and Lewin-type Clausen representations, but it has **no known evaluation** in terms of π, logarithms and Catalan's constant G (the known closed-form points of Ti_2 are $\tan\frac{\pi}{4}=1$, $\tan\frac{\pi}{12} = 2-\sqrt{3}$, etc., and $\frac{1}{2}$ is not of that form). As experimental support, an integer-relation (PSLQ) search at 250-digit precision against the basis

$$\{G,\ \pi\ln 2,\ \pi\ln 3,\ \pi\ln 5,\ \arctan\tfrac{1}{2}\ln 2,\ \arctan\tfrac{1}{2}\ln 5,\ \pi^2,\ \arctan^2\tfrac{1}{2},\ \pi\arctan\tfrac{1}{2}\}$$

with coefficients up to 10^{10} found no relation (a candidate produced at 60 digits failed at 250 digits, residual $\sim 10^{-45}$, i.e. spurious). So the answer above is in lowest terms in the accepted sense.

Numerical verification

All computations with mpmath (working precision up to 250 digits).

- Direct quadrature, `mp.quad` on $[\pi/2,\pi]$ at `mp.mp.dps = 250`:

$$I_{\text{num}} = 0.305663655624456784287951111999948202903953469297 06\ldots$$

- Closed form $\frac{\pi}{8}\ln\frac{81}{20} - \frac{1}{2}\operatorname{Ti}_2(\frac{1}{2})$ with $\operatorname{Ti}_2(\frac{1}{2}) = \Im\operatorname{Li}_2(i/2)$ computed three independent ways (mpmath `polylog`, the alternating series $\sum_k \frac{(-1)^k}{(2k+1)^2 2^{2k+1}}$, and quadrature of $\int_0^{1/2}\frac{\arctan t}{t}dt$ — all agree to full precision):

$$I_{\text{cf}} = 0.30566365562445678428795111199994820290395346929706\ldots$$

- $|I_{\text{num}} - I_{\text{cf}}| \approx 8.7 \times 10^{-252}$: **agreement to about 250 significant digits.**
- Sub-identity checks: $\int_{\pi/2}^{\pi} \ln(5 - 4\cos x)\, dx = \pi \ln 2 + 2\,\mathrm{Ti}_2(\frac{1}{2})$ verified to 3×10^{-61} at 60 digits; the classical full-range value $\frac{\pi}{2} \ln \frac{3}{2}$ verified to machine-zero at 60 digits.

Reference values (50 significant digits):

$$\mathrm{Ti}_2(\tfrac{1}{2}) = 0.48722235829452235711023449769333797060906604030860,$$

$$I = 0.30566365562445678428795111199994820290395346929706.$$

Notes

- The derivation is complete and rigorous: the only limit interchange (term-by-term integration of the log Fourier series) is justified by uniform convergence via the Weierstrass M-test on $|z| = \frac{1}{2} < 1$, and an alternative series-free justification via the antiderivative $\Im\,\mathrm{Li}_2(\frac{1}{2}e^{ix})$ is included. The integration by parts is between C^1 functions on a compact interval with a strictly positive denominator.
- The answer necessarily involves one non-elementary constant, $\mathrm{Ti}_2(1/2) = \Im\,\mathrm{Li}_2(i/2)$. It is *believed* (and supported by the 250-digit PSLQ search reported above) that this constant is not expressible via π, ln, and Catalan's constant; that impossibility is of course not proven — this is the only caveat, and it concerns the form, not the correctness, of the result.
- Equivalent presentations: $I = \frac{\pi}{2} \ln 3 - \frac{\pi}{4} \ln 2 - \frac{\pi}{8} \ln 5 - \frac{1}{2} \Im\,\mathrm{Li}_2(i/2)$, or with $\mathrm{Ti}_2(2)$ via $\mathrm{Ti}_2(2) = \mathrm{Ti}_2(1/2) + \frac{\pi}{2} \ln 2$, giving $I = \frac{\pi}{8} \ln \frac{81 \cdot 4}{20} - \ldots$ hmm — concretely $I = \frac{\pi}{8} \ln \frac{81}{5} + \frac{\pi}{8} \ln \frac{4}{4}$; cleanly: $I = \frac{\pi}{8} \ln \frac{81}{5} - \frac{\pi}{8} \ln 4 + \frac{\pi}{4} \ln 2 - \frac{1}{2}\,\mathrm{Ti}_2(2) + \frac{\pi}{4} \ln 2$ reduces to $I = \frac{\pi}{8} \ln \frac{324}{5} - \frac{1}{2}\,\mathrm{Ti}_2(2)$.