

Cleo Bench Problem 5

$$\text{Integral } \int_0^{\pi/2} \arctan^2\left(\frac{\sin x}{\sqrt{3} + \cos x}\right) dx$$

Derivation by Claude (Fable 5), closed-book*

July 2026

Problem

Evaluate in closed form

$$I = \int_0^{\pi/2} \arctan^2\left(\frac{\sin x}{\sqrt{3} + \cos x}\right) dx, \quad I \approx 0.15637139137571170123083760 \dots$$

Result

Throughout, Li_s is the principal polylogarithm, $q = \frac{1}{\sqrt{3}}$, and

$$\mu = \frac{e^{i\pi/6}}{\sqrt{3}} = \frac{3 + i\sqrt{3}}{6}, \quad iq = \frac{i}{\sqrt{3}}.$$

Main result (proved unconditionally below):

$$I = \text{Im Li}_3\left(\frac{i}{\sqrt{3}}\right) - \text{Im Li}_3\left(\frac{3 + i\sqrt{3}}{6}\right) + \frac{\pi}{4} \text{Li}_2\left(\frac{1}{3}\right) - \frac{11\pi^3}{648} + \frac{\pi \ln^2 3}{24}$$

Equivalent minimal form. The two trilogarithm constants satisfy the ladder relation

$$(\star) \quad \text{Im Li}_3\left(\frac{3 + i\sqrt{3}}{6}\right) = \frac{4}{5} \text{Im Li}_3\left(\frac{i}{\sqrt{3}}\right) + \frac{\pi \ln^2 3}{240} - \frac{29\pi^3}{6480},$$

which I have verified to 300 decimal digits (but do **not** prove here; see Notes). Substituting (\star) gives the compact equivalent form

$$I = \frac{\pi}{4} \text{Li}_2\left(\frac{1}{3}\right) + \frac{1}{5} \text{Im Li}_3\left(\frac{i}{\sqrt{3}}\right) - \frac{\pi^3}{80} + \frac{3\pi \ln^2 3}{80},$$

where $\text{Im Li}_3(iy) = \text{Ti}_3(y) = \sum_{n \geq 0} \frac{(-1)^n y^{2n+1}}{(2n+1)^3}$ is the inverse tangent integral of order 3.

Both displayed forms agree with the direct numerical value of I to more than 300 significant digits.

*Problem originally posed on Mathematics Stack Exchange ([question 1595389](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Derivation

0. Notation and plan

Let $q = 1/\sqrt{3}$ and

$$L(x) = \ln(1 + qe^{ix}) \quad (\text{principal branch}), \quad u(x) = \operatorname{Re} L, \quad v(x) = \operatorname{Im} L.$$

For $x \in [0, \pi/2]$, $1 + qe^{ix}$ lies in the right half-plane, so

$$v(x) = \arg(1 + qe^{ix}) = \arctan \frac{q \sin x}{1 + q \cos x} = \arctan \frac{\sin x}{\sqrt{3} + \cos x},$$

hence $I = \int_0^{\pi/2} v^2 dx$. From $L = u + iv$ we get the pointwise identity

$$v^2 = \frac{1}{2} (|L|^2 - \operatorname{Re} L^2),$$

so

$$I = \frac{1}{2} \int_0^{\pi/2} |L|^2 dx - \frac{1}{2} \operatorname{Re} \int_0^{\pi/2} L^2 dx. \quad (0.1)$$

Both integrals will be computed in closed form by contour methods.

Special points used throughout (all identities are trivial arithmetic in $\mathbb{Q}(i, \sqrt{3})$, and each was also checked numerically):

$$\begin{aligned} \nu &= 1 + \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}} e^{i\pi/6}, & p_4 &= 1 + i\sqrt{3} = 2e^{i\pi/3}, & \mu &= \frac{e^{i\pi/6}}{\sqrt{3}}, \\ w_0 &= \frac{1}{2} e^{i\pi/3}, & w_1 &= \frac{\sqrt{3}}{2} e^{i\pi/6} = \frac{3+i\sqrt{3}}{4}, \end{aligned}$$

$$\begin{aligned} 1 - iq &= \bar{\nu}, & 1 - \frac{1}{iq} &= p_4, & \frac{1}{\nu} &= \bar{w}_1, & \frac{1}{p_4} &= \bar{w}_0, \\ 1 - w_1 &= \bar{w}_0, & -\frac{iq}{1-iq} &= \bar{w}_0, & -\frac{\mu}{1-\mu} &= e^{-2\pi i/3}, & 1 - \mu &= \bar{\mu}. \end{aligned}$$

Principal logarithms (each verified): with $\ell_2 = \ln 2$, $\ell_3 = \ln 3$,

$$\begin{aligned} \ln(-qi) &= -\frac{\ell_3}{2} - \frac{i\pi}{2}, & \ln \nu &= \ell_2 - \frac{\ell_3}{2} + \frac{i\pi}{6}, & \ln(-\sqrt{3}i) &= \frac{\ell_3}{2} - \frac{i\pi}{2}, & \ln p_4 &= \ell_2 + \frac{i\pi}{3}, \\ \ln \mu &= -\frac{\ell_3}{2} + \frac{i\pi}{6}, & \ln \bar{\mu} &= -\frac{\ell_3}{2} - \frac{i\pi}{6}, & \ln(iq) &= -\frac{\ell_3}{2} + \frac{i\pi}{2}, & \ln(1 - iq) &= \ell_2 - \frac{\ell_3}{2} - \frac{i\pi}{6}. \end{aligned}$$

1. Two auxiliary primitives

Lemma 1 (the Φ -integral). For w in the cut plane $\mathbb{C} \setminus [1, \infty)$ let

$$\Phi(w) = \int_0^w \frac{\ln^2(1-s)}{s} ds \quad (\text{integral along the segment; the integrand is } O(s) \text{ at } 0).$$

Then, whenever the segment $[1, 1-w]$ (the image of the integration segment under $s \mapsto 1-s$) avoids $(-\infty, 0] \cup [1, \infty)$ except for its endpoint 1,

$$\Phi(w) = 2\zeta(3) + \ln w \ln^2(1-w) + 2 \ln(1-w) \operatorname{Li}_2(1-w) - 2 \operatorname{Li}_3(1-w). \quad (1.1)$$

Proof. Let

$$G(s) = -\ln(1-s) \ln^2 s - 2 \ln s \operatorname{Li}_2(s) + 2 \operatorname{Li}_3(s).$$

G is analytic on $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ and, using $\operatorname{Li}'_2(s) = -\ln(1-s)/s$, $\operatorname{Li}'_3(s) = \operatorname{Li}_2(s)/s$,

$$G'(s) = \frac{\ln^2 s}{1-s} - \frac{2 \ln s \ln(1-s)}{s} + \frac{2 \ln s \ln(1-s)}{s} - \frac{2 \operatorname{Li}_2(s)}{s} + \frac{2 \operatorname{Li}_2(s)}{s} = \frac{\ln^2 s}{1-s}.$$

Substituting $s = 1 - \sigma$ in Φ ,

$$\Phi(w) = \int_{1-w}^1 \frac{\ln^2 \sigma}{1-\sigma} d\sigma = G(1) - G(1-w) = 2\zeta(3) - G(1-w),$$

which is (1.1); $G(1) = 2\text{Li}_3(1) = 2\zeta(3)$ because $\ln(1-s)\ln^2 s \rightarrow 0$ and $\ln s \text{Li}_2(s) \rightarrow 0$ as $s \rightarrow 1$ inside the domain. The hypothesis on the segment guarantees Newton–Leibniz applies with principal branches. \square

We will use (1.1) at $w = -qi$ (then $1-w = \nu$; the segment $[1, \nu]$ lies in the closed upper right region, off both cuts) and at $w = -\sqrt{3}i$ (then $1-w = p_4$; segment $[1, p_4]$ likewise admissible). At $w = -q$ and $w = -\sqrt{3}$ we only need that $\Phi(w) \in \mathbb{R}$, which is clear from the real integrand.

Lemma 2 (log-weighted primitive). On the closed first quadrant minus the origin,

$$P_1(z) = \text{Li}_3(-qz) - \ln z \text{Li}_2(-qz) \quad \text{satisfies} \quad P_1'(z) = \frac{\ln z \ln(1+qz)}{z}.$$

Proof. $\frac{d}{dz} \text{Li}_3(-qz) = \text{Li}_2(-qz)/z$ and $\frac{d}{dz} \text{Li}_2(-qz) = -\ln(1+qz)/z$; the product rule gives the claim. All functions are analytic near the quarter arc since $-qz \notin [1, \infty)$ there and $z \notin (-\infty, 0]$. \square

2. The square integral $\int L^2$

With $z = e^{ix}$, $dx = dz/(iz)$, and the integrand $h(t) = \ln^2(1+qt)/t$ analytic on the closed quarter disk minus $\{0\}$ (its branch points $-\sqrt{3}$ and 0 are outside/removable: $h(t) = O(t)$ at 0), Cauchy's theorem on the quarter-disk contour (segment $[0, 1]$ + arc + segment $[i, 0]$, with an indent at 0 whose contribution is $O(\varepsilon)$) gives

$$\int_0^{\pi/2} L^2 dx = \frac{1}{i} \int_{\text{arc } 1 \rightarrow i} \frac{\ln^2(1+qz)}{z} dz = \frac{1}{i} [\Phi(-qi) - \Phi(-q)],$$

because $\frac{d}{dz} \Phi(-qz) = \ln^2(1+qz)/z$ (same computation as in Lemma 2). Since $\Phi(-q) \in \mathbb{R}$,

$$\text{Re} \int_0^{\pi/2} L^2 dx = \text{Im} \Phi(-qi). \quad (2.1)$$

3. The modulus-square integral $\int |L|^2$

For $x \in [0, \pi/2]$,

$$1 + qe^{-ix} = e^{-ix} (e^{ix} + q) \implies \overline{L(x)} = \ln(q + e^{ix}) - ix, \quad (3.1)$$

with principal branches: the two sides differ by a continuous integer multiple of $2\pi i$ which vanishes at $x = 0$ (identity (3.1) was also verified numerically along the whole arc). Hence

$$\int_0^{\pi/2} |L|^2 dx = M_2 - M_1, \quad M_1 := i \int_0^{\pi/2} xL dx, \quad M_2 := \int_0^{\pi/2} L(x) \ln(q + e^{ix}) dx.$$

M_1 . In contour form $M_1 = \frac{1}{i} \int_{\text{arc}} \ln(1+qz) \ln z \frac{dz}{z}$; by Lemma 2 and the same Cauchy argument (integrand $\sim q \ln t$ at 0 , indent contribution $O(\varepsilon \ln \varepsilon) \rightarrow 0$),

$$M_1 = \frac{1}{i} [P_1(i) - P_1(1)] = \frac{1}{i} \left[\text{Li}_3(-qi) - \frac{i\pi}{2} \text{Li}_2(-qi) - \text{Li}_3(-q) \right].$$

Since $\text{Li}_s(-qi) = \overline{\text{Li}_s(qi)}$ and $\text{Li}_3(-q) \in \mathbb{R}$,

$$\text{Re} M_1 = -\text{Im} \text{Li}_3(iq) - \frac{\pi}{2} \text{Re} \text{Li}_2(iq). \quad (3.2)$$

M_2 . Again by Cauchy, $M_2 = \frac{1}{i}[K(i) - K(1)]$ where

$$K(z) = \int_0^z \frac{\ln(1+qt)\ln(q+t)}{t} dt \quad (\text{segments from } 0).$$

On the closed first quadrant $\ln(q+t) = \ln q + \ln(1+t/q)$ (both factors lie in the right half-plane; verified along all paths), so

$$K(z) = -\ln q \operatorname{Li}_2(-qz) + N(z), \quad N(z) = \int_0^z \frac{\ln(1+qt)\ln(1+t/q)}{t} dt.$$

Because both logarithms have arguments in the right half-plane, with

$$\varphi(t) = \frac{1+qt}{1+t/q} \quad \text{we have} \quad \ln \varphi(t) = \ln(1+qt) - \ln(1+t/q)$$

(principal branches, verified), and the algebraic identity $XY = \frac{1}{2}[X^2 + Y^2 - (X - Y)^2]$ gives

$$N(z) = \frac{1}{2}\Phi(-qz) + \frac{1}{2}\Phi(-z/q) - \frac{1}{2}R(z), \quad R(z) = \int_0^z \frac{\ln^2 \varphi(t)}{t} dt. \quad (3.3)$$

Evaluation of $R(i)$. Substitute $s = \varphi(t)$. Solving, $t = \frac{q(1-s)}{s-q^2}$, so

$$\frac{dt}{t} = \left[\frac{1}{s-1} - \frac{1}{s-c} \right] ds, \quad c = q^2 = \frac{1}{3}.$$

As t runs over the segment $[0, i]$, s runs along a path from 1 to

$$S := \varphi(i) = \frac{1+qi}{1+i/q} = \frac{(1+qi)(1-i\sqrt{3})}{4} = \frac{1}{2} \left(1 - \frac{i}{\sqrt{3}} \right) = \bar{\mu}$$

(direct computation), and a numerical trace of the path shows it stays in $\{\frac{1}{2} \leq \operatorname{Re} s \leq 1, -\frac{1}{3} \leq \operatorname{Im} s \leq 0\}$, i.e. inside $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ except for the initial endpoint $s = 1$, where the integrand $\ln^2 s/(s-1)$ is bounded. Hence, with G from Lemma 1,

$$\int_1^{\bar{\mu}} \frac{\ln^2 s}{s-1} ds = 2\zeta(3) - G(\bar{\mu}).$$

For the second part substitute $\tau = c/s = 1/(3s)$ (so $\ln s = \ln c - \ln \tau$, valid since $c > 0$ and τ stays off $(-\infty, 0]$): the τ -path runs from $\frac{1}{3}$ to $1/(3\bar{\mu}) = \mu$ inside $\{\frac{1}{3} \leq \operatorname{Re} \tau \leq \frac{1}{2}, 0 \leq \operatorname{Im} \tau \leq 0.29\}$, again inside the cut plane, and

$$\begin{aligned} \int_1^{\bar{\mu}} \frac{\ln^2 s}{s-c} ds &= \ln^2 c \left[\ln(1-\tau) - \ln \tau \right]_{1/3}^{\mu} + 2 \ln c \left[\frac{1}{2} \ln^2 \tau + \operatorname{Li}_2(1-\tau) \right]_{1/3}^{\mu} \\ &\quad - \left[\frac{1}{3} \ln^3 \tau + G(\tau) \right]_{1/3}^{\mu}, \end{aligned}$$

using $\frac{ds}{s-c} = -\left[\frac{1}{\tau} + \frac{1}{1-\tau} \right] d\tau$ and the primitives $\int \frac{\ln \tau}{1-\tau} d\tau = \operatorname{Li}_2(1-\tau)$, $\int \frac{\ln^2 \tau}{1-\tau} d\tau = G(\tau)$. Combining $(1-\mu = \bar{\mu}, \ln \frac{1}{3} = -\ell_3, \ln \frac{2}{3} = \ell_2 - \ell_3)$:

$$R(i) = 2\zeta(3) - G(\bar{\mu}) - \left[\ell_3^2 A_0 - 2\ell_3 A_1 + A_2 \right], \quad (3.4)$$

$$A_0 = (\ln \bar{\mu} - \ln \mu) - \ell_2, \quad A_1 = \left(\frac{1}{2} \ln^2 \mu + \operatorname{Li}_2(\bar{\mu}) \right) - \left(\frac{1}{2} \ell_3^2 + \operatorname{Li}_2\left(\frac{2}{3}\right) \right),$$

$$A_2 = -\left(\frac{1}{3} \ln^3 \mu + G(\mu) \right) + \left(-\frac{1}{3} \ell_3^3 + G\left(\frac{1}{3}\right) \right).$$

(Every displayed reduction in this section — the closed forms for $R(z)$, $N(z)$, K , M_1 , M_2 , $\int L^2$, and the final assembly — was verified numerically to 60 digits at $z = 1$, $z = i$ and at a generic complex point.)

4. Assembly

By (0.1), (2.1), (3.2):

$$I = \frac{1}{2} \left[\operatorname{Im} K(i) + \operatorname{Im} \operatorname{Li}_3(iq) + \frac{\pi}{2} \operatorname{Re} \operatorname{Li}_2(iq) - \operatorname{Im} \Phi(-qi) \right],$$

$$\operatorname{Im} K(i) = \frac{\ell_3}{2} \operatorname{Im} \operatorname{Li}_2(-qi) + \frac{1}{2} \operatorname{Im} \Phi(-qi) + \frac{1}{2} \operatorname{Im} \Phi(-\sqrt{3}i) - \frac{1}{2} \operatorname{Im} R(i),$$

where $\operatorname{Im} \operatorname{Li}_2(-qi) = -\operatorname{Im} \operatorname{Li}_2(iq)$. All *real* quantities ($\Phi(-q)$, $K(1)$, $G(\frac{1}{3})$, $\operatorname{Li}_2(\frac{2}{3})$, $\zeta(3)$, ...) drop out of the imaginary parts. Expanding $\operatorname{Im} \Phi(-qi)$, $\operatorname{Im} \Phi(-\sqrt{3}i)$ by (1.1) (arguments ν , p_4), and $\operatorname{Im} R(i)$ by (3.4) with

$$G(\bar{\mu}) = -\ln \mu \ln^2 \bar{\mu} - 2 \ln \bar{\mu} \operatorname{Li}_2(\bar{\mu}) + 2 \operatorname{Li}_3(\bar{\mu}), \quad G(\mu) = -\ln \bar{\mu} \ln^2 \mu - 2 \ln \mu \operatorname{Li}_2(\mu) + 2 \operatorname{Li}_3(\mu),$$

and inserting the principal logarithms listed in §0, a routine (computer-verified) expansion yields the exact intermediate formula

$$\begin{aligned} I = & \frac{17\pi^3}{1296} - \frac{\pi\ell_3^2}{48} + \frac{\pi}{6} \operatorname{Re} \operatorname{Li}_2(\mu) - \frac{\pi}{12} \operatorname{Re} \operatorname{Li}_2(\nu) + \frac{\pi}{4} \operatorname{Re} \operatorname{Li}_2(iq) + \frac{\pi}{6} \operatorname{Re} \operatorname{Li}_2(p_4) \\ & - \frac{\ell_2}{2} \operatorname{Im} \operatorname{Li}_2(\nu) + \frac{\ell_2}{2} \operatorname{Im} \operatorname{Li}_2(p_4) + \frac{\ell_3}{4} \operatorname{Im} \operatorname{Li}_2(\nu) - \frac{\ell_3}{4} \operatorname{Im} \operatorname{Li}_2(iq) \\ & - \operatorname{Im} \operatorname{Li}_3(\mu) + \frac{1}{2} \operatorname{Im} \operatorname{Li}_3(\nu) + \frac{1}{2} \operatorname{Im} \operatorname{Li}_3(iq) - \frac{1}{2} \operatorname{Im} \operatorname{Li}_3(p_4). \end{aligned} \quad (4.1)$$

(Formula (4.1) reproduces the numerical value of I to 60 digits.)

It remains to evaluate the dilogarithms and to relate the three trilogarithms $\operatorname{Im} \operatorname{Li}_3(\nu)$, $\operatorname{Im} \operatorname{Li}_3(p_4)$, $\operatorname{Im} \operatorname{Li}_3(iq)$.

5. Dilogarithm lemmas

All classical; each is proved by differentiating both sides (the derivatives match using only $\operatorname{Li}'_2(z) = -\ln(1-z)/z$) and fixing the constant at one point, and each is then extended to the needed complex point by the identity principle: in every instance below both sides are analytic on the upper half-plane \mathbb{H} (the relevant compositions avoid the cuts $[1, \infty)$, $(-\infty, 0]$ there — e.g. $z \in \mathbb{H} \Rightarrow 1-z$ in the lower half-plane, $-z/(1-z)$ in the lower half-plane, $1/z$ in the lower half-plane), and they agree on a real interval adjacent to \mathbb{H} or along a curve reaching it; every instance was verified numerically to 300 digits at the exact point of use.

(L1) Reflection. $\operatorname{Li}_2(z) + \operatorname{Li}_2(1-z) = \frac{\pi^2}{6} - \ln z \ln(1-z)$. (At $z = \frac{1}{2}$: $\operatorname{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$.)

(L2) Landen. $\operatorname{Li}_2(z) + \operatorname{Li}_2(\frac{-z}{1-z}) = -\frac{1}{2} \ln^2(1-z)$.

(L3) Inversion. $\operatorname{Li}_2(z) + \operatorname{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(-z)$ for $z \notin [0, \infty)$.

(L4) Duplication. $\operatorname{Li}_2(z^2) = 2[\operatorname{Li}_2(z) + \operatorname{Li}_2(-z)]$.

(L5) Unit circle. For $0 \leq \theta \leq 2\pi$: $\operatorname{Re} \operatorname{Li}_2(e^{i\theta}) = \frac{\pi^2}{6} - \frac{\pi\theta}{2} + \frac{\theta^2}{4}$ and $\operatorname{Im} \operatorname{Li}_2(e^{i\theta}) = \operatorname{Cl}_2(\theta)$ (Fourier series of the Bernoulli polynomial; $\operatorname{Cl}_2 =$ Clausen function). Together with the Clausen duplication $\operatorname{Cl}_2(2\theta) = 2 \operatorname{Cl}_2(\theta) - 2 \operatorname{Cl}_2(\pi - \theta)$ (from $\sin 2\theta = 2 \sin \theta \cos \theta$), which at $\theta = \pi/3$ gives $\operatorname{Cl}_2(2\pi/3) = \frac{2}{3} \operatorname{Cl}_2(\pi/3)$, one gets

$$\operatorname{Li}_2(e^{-2\pi i/3}) = -\frac{\pi^2}{18} - \frac{2i}{3} \operatorname{Cl}_2\left(\frac{\pi}{3}\right).$$

(L6) The point μ . Landen (L2) at $z = \mu$ (note $-\mu/(1-\mu) = -\mu/\bar{\mu} = e^{-2\pi i/3}$, $1-\mu = \bar{\mu}$) plus (L5):

$$\operatorname{Li}_2(\mu) = -\frac{1}{2} \ln^2 \bar{\mu} - \operatorname{Li}_2(e^{-2\pi i/3}) \implies \operatorname{Re} \operatorname{Li}_2(\mu) = \frac{5\pi^2}{72} - \frac{\ell_3^2}{8}.$$

(L7) The point iq . *Real part.* By (L4) at $z = iq$: $\operatorname{Re} \operatorname{Li}_2(iq) = \frac{1}{2} [\operatorname{Li}_2(iq) + \operatorname{Li}_2(-iq)] = \frac{1}{4} \operatorname{Li}_2(-\frac{1}{3})$; and the two-Landen/duplication chain

$$\operatorname{Li}_2(\frac{1}{3}) + \operatorname{Li}_2(-\frac{1}{2}) = -\frac{1}{2} \ln^2 \frac{2}{3}, \quad \operatorname{Li}_2(-\frac{1}{3}) + \operatorname{Li}_2(\frac{1}{4}) = -\frac{1}{2} \ln^2 \frac{4}{3}, \quad \operatorname{Li}_2(\frac{1}{4}) = 2 \operatorname{Li}_2(\frac{1}{2}) + 2 \operatorname{Li}_2(-\frac{1}{2})$$

with $\operatorname{Li}_2(\frac{1}{2})$ from (L1) gives the classical relation

$$\operatorname{Li}_2\left(-\frac{1}{3}\right) = 2 \operatorname{Li}_2\left(\frac{1}{3}\right) + \frac{\ln^2 3}{2} - \frac{\pi^2}{6} \implies \operatorname{Re} \operatorname{Li}_2(iq) = \frac{1}{2} \operatorname{Li}_2\left(\frac{1}{3}\right) + \frac{\ell_3^2}{8} - \frac{\pi^2}{24}.$$

Imaginary part. $\operatorname{Im} \operatorname{Li}_2(iq) = \operatorname{Ti}_2(q)$, and the classical formula

$$\operatorname{Ti}_2(\tan \theta) = \theta \ln \tan \theta + \frac{1}{2} \operatorname{Cl}_2(2\theta) + \frac{1}{2} \operatorname{Cl}_2(\pi - 2\theta) \quad (0 < \theta < \pi/2)$$

(proved by differentiating in θ : both sides have derivative $2\theta/\sin 2\theta$, and both vanish as $\theta \rightarrow 0^+$) at $\theta = \pi/6$, with $\operatorname{Cl}_2(2\pi/3) = \frac{2}{3} \operatorname{Cl}_2(\pi/3)$, gives

$$\operatorname{Im} \operatorname{Li}_2(iq) = \operatorname{Ti}_2\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{6} \operatorname{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi \ell_3}{12}.$$

(L8) The points ν and p_4 . Landen (L2) at $z = iq$ (note $-iq/(1-iq) = \bar{w}_0$, $1-iq = \bar{\nu}$):

$$\operatorname{Li}_2(\bar{w}_0) = -\frac{1}{2} \ln^2 \bar{\nu} - \operatorname{Li}_2(iq).$$

Reflection (L1) at $z = w_1$ ($1-w_1 = \bar{w}_0$): $\operatorname{Li}_2(w_1) = \frac{\pi^2}{6} - \ln w_1 \ln \bar{w}_0 - \operatorname{Li}_2(\bar{w}_0)$. Inversion (L3) at $z = \nu$ ($1/\nu = \bar{w}_1$, $\ln(-\nu) = \ell_2 - \frac{\ell_3}{2} - \frac{5\pi i}{6}$) and at $z = p_4$ ($1/p_4 = \bar{w}_0$, $\ln(-p_4) = \ell_2 - \frac{2\pi i}{3}$) then yield, after inserting (L7):

$$\operatorname{Li}_2(\nu) = -\frac{1}{4} \operatorname{Li}_2(-\frac{1}{3}) + \frac{\pi^2}{12} + \frac{\ell_2 \ell_3}{2} - \frac{\ell_3^2}{4} + i\left(\frac{5}{6} \operatorname{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi \ell_2}{2} - \frac{\pi \ell_3}{4}\right),$$

$$\operatorname{Li}_2(p_4) = \frac{1}{4} \operatorname{Li}_2(-\frac{1}{3}) + \frac{\pi^2}{24} - \frac{\ell_2 \ell_3}{2} + \frac{\ell_3^2}{8} + i\left(\frac{5}{6} \operatorname{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi \ell_2}{2}\right).$$

(Each was checked to 60 digits.)

6. The trilogarithm input: Kummer's three-term identity at $z = iq$

Lemma 9. For z in the upper half-plane (and on $(0, 1)$),

$$\operatorname{Li}_3(z) + \operatorname{Li}_3(1-z) + \operatorname{Li}_3\left(1 - \frac{1}{z}\right) = \zeta(3) + \frac{\ln^3 z}{6} + \frac{\pi^2}{6} \ln z - \frac{1}{2} \ln^2 z \ln(1-z). \quad (6.1)$$

Proof. For $z \in (0, 1)$ differentiate. Using $\operatorname{Li}'_3(w) = \operatorname{Li}_2(w)/w$ and the Landen identity (L2) in the form $\operatorname{Li}_2(1 - \frac{1}{z}) = -\frac{1}{2} \ln^2 z - \operatorname{Li}_2(1-z)$,

$$\frac{d}{dz} \text{LHS} = \frac{\operatorname{Li}_2(z)}{z} - \frac{\operatorname{Li}_2(1-z)}{1-z} + \frac{\operatorname{Li}_2(1 - \frac{1}{z})}{z(z-1)} = \frac{\operatorname{Li}_2(z) + \operatorname{Li}_2(1-z)}{z} - \frac{\ln^2 z}{2} \left(\frac{1}{z-1} - \frac{1}{z}\right),$$

where we expanded $\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$. By reflection (L1) this equals

$$\frac{\pi^2}{6z} - \frac{\ln z \ln(1-z)}{z} + \frac{\ln^2 z}{2(1-z)} + \frac{\ln^2 z}{2z} = \frac{d}{dz} \text{RHS}.$$

As $z \rightarrow 1^-$ both sides tend to $\zeta(3)$. Hence (6.1) holds on $(0, 1)$. For $z \in \mathbb{H}$: $1 - z$ and $1 - 1/z$ avoid the cut $[1, \infty)$ ($1 - z$ is in the lower half-plane; $\text{Im}(1 - 1/z) = \text{Im}(-1/z) > 0$), and $\ln z$, $\ln(1 - z)$ are principal-analytic, so both sides are analytic on $\mathbb{H} \cup (0, 1)$ -neighbourhood and agree on $(0, 1)$; by the identity principle they agree on \mathbb{H} . (Verified at $z = iq$ to 300 digits.) \square

At $z = iq$ we have $1 - z = \bar{\nu}$ and $1 - 1/z = 1 + i\sqrt{3} = p_4$, so taking imaginary parts of (6.1) (recall $\text{Li}_3(\bar{\nu}) = \overline{\text{Li}_3(\nu)}$), with $\ln(iq) = -\frac{\ell_3}{2} + \frac{i\pi}{2}$, $\ln(1 - iq) = \ell_2 - \frac{\ell_3}{2} - \frac{i\pi}{6}$:

$$\text{Im Li}_3(iq) - \text{Im Li}_3(\nu) + \text{Im Li}_3(p_4) = \frac{\pi(\pi^2 + 6\ell_2\ell_3 - \ell_3^2)}{24}. \quad (6.2)$$

Consequently, in (4.1) the combination $\frac{1}{2} \text{Im Li}_3(\nu) - \frac{1}{2} \text{Im Li}_3(p_4)$ equals $\frac{1}{2} \text{Im Li}_3(iq) - \frac{\pi(\pi^2 + 6\ell_2\ell_3 - \ell_3^2)}{48}$: the constants $\text{Im Li}_3(\nu)$, $\text{Im Li}_3(p_4)$ disappear.

7. Final collapse

Substituting into (4.1) the exact values from (L6), (L7), (L8) and eliminating $\text{Im Li}_3(\nu)$, $\text{Im Li}_3(p_4)$ by (6.2), all terms containing $\text{Cl}_2(\pi/3)$, ℓ_2^2 , $\ell_2\ell_3$ cancel identically, and with $\text{Li}_2(-\frac{1}{3}) = 2\text{Li}_2(\frac{1}{3}) + \frac{\ell_3^2}{2} - \frac{\pi^2}{6}$ one obtains

$$I = \text{Im Li}_3(iq) - \text{Im Li}_3(\mu) + \frac{\pi}{4} \text{Li}_2\left(\frac{1}{3}\right) - \frac{11\pi^3}{648} + \frac{\pi\ell_3^2}{24} \quad (7.1)$$

The elementary bookkeeping from (4.1) to (7.1) was performed and verified in exact arithmetic with a computer algebra system (sympy), and (7.1) agrees with the direct numerical value of I to 310 significant digits.

8. The ladder (\star) and the minimal form

PSLQ applied to high-precision values reveals the (numerically certain, 300-digit) relation

$$(\star) \quad \text{Im Li}_3(\mu) = \frac{4}{5} \text{Im Li}_3(iq) + \frac{\pi\ell_3^2}{240} - \frac{29\pi^3}{6480}.$$

Substituting (\star) into (7.1) gives the minimal form quoted in the Result:

$$I = \frac{\pi}{4} \text{Li}_2\left(\frac{1}{3}\right) + \frac{1}{5} \text{Im Li}_3\left(\frac{i}{\sqrt{3}}\right) - \frac{\pi^3}{80} + \frac{3\pi\ell_3^2}{80}. \quad (8.1)$$

I emphasize: (7.1) is proved unconditionally above; (8.1) is equivalent to it modulo (\star), which I verify numerically to 300 digits but do not prove. Relation (\star) is a genuine “trilogarithmic ladder” at the base μ (which satisfies $3\mu^2 - 3\mu + 1 = 0$, $|\mu|^2 = \frac{1}{3}$, $1 - \mu = \bar{\mu}$): an exhaustive check of all Kummer/Landen/inversion/duplication/triplication instances whose arguments stay in the multiplicative lattice $\{2^a 3^{b/2} e^{ik\pi/6}\}$ shows the single-variable functional equations do *not* suffice to derive it (they yield only (6.2) and real-part relations), so a proof requires a two-variable trilogarithm equation (Kummer’s equation) or equivalent machinery, which is not carried out here.

Numerical verification

All computations with `mpmath` at working precision up to 310 digits.

- Direct evaluation (tanh-sinh quadrature of the smooth integrand, `mp.quad`, `dps = 310`):

$$I = 0.15637139137571170123083760326663152202040959779133939842807852631660402\dots$$

matching the value quoted in the problem statement.

- Main closed form (7.1): $|I_{\text{closed}} - I_{\text{quad}}| < 10^{-311}$ — agreement to ≈ 310 significant digits.
- Minimal form (8.1): same agreement ($< 10^{-311}$).
- Every intermediate identity was verified numerically at its exact point of use: contour reductions for $\int L^2$, M_1 , M_2 , N , R (60 digits, three different endpoints); the fourteen-term intermediate formula (4.1) (60 digits); lemmas (L1)–(L8), Lemma 9, the principal-log values, and the point coincidences (all $\leq 10^{-300}$ residuals at 310 dps); the path traces for the s - and τ -substitutions (1000 sample points each, no cut crossings); the branch identities (3.1), $\ln(q+t) = \ln q + \ln(1+t/q)$, $\ln \varphi = \ln(1+qt) - \ln(1+t/q)$ along the full paths.
- The ladder (\star): residual $< 10^{-300}$ (PSLQ at 260 digits, small coefficients; independently re-verified at 310 digits).

Notes

1. **What is fully proved.** The closed form (7.1) — equivalently the boxed Result — is derived completely: Fourier/contour reduction, explicit polylogarithmic primitives with path/branch control, classical one-variable dilogarithm identities (each proved by differentiation + identity principle), and Kummer’s three-term trilogarithm identity (proved in Lemma 9). The final coefficient collapse was done in exact rational arithmetic.
2. **The one unproved step for the minimal form.** The reduction from the two-trilogarithm form (7.1) to the one-trilogarithm form (8.1) uses the ladder (\star), verified to 300 digits but not proved here. Both forms are exact closed forms of I ; (7.1) does not depend on (\star).
3. The constants $\text{Im Li}_3(i/\sqrt{3}) = \text{Ti}_3(1/\sqrt{3})$ and $\text{Li}_2(1/3)$ appear to be irreducible: PSLQ finds no relation of either against $\{\pi^3, \pi \ln^2 2, \pi \ln^2 3, \pi \ln 2 \ln 3, \text{Cl}_2(\pi/3) \ln 2, \text{Cl}_2(\pi/3) \ln 3, G \ln 2, G \ln 3, \pi \text{Li}_2(1/3)\}$ with coefficients up to 10^{12} .
4. Useful byproducts of the derivation (all proved): $\int_0^\pi \arctan^2 \frac{\sin x}{\sqrt{3+\cos x}} dx = \frac{\pi}{2} \text{Li}_2(\frac{1}{3})$ (Parseval), $\text{Im Li}_2(1 + \frac{i}{\sqrt{3}}) = \frac{5}{6} \text{Cl}_2(\frac{\pi}{3}) + \frac{\pi \ln 2}{2} - \frac{\pi \ln 3}{4}$, and $\text{Ti}_2(\frac{1}{\sqrt{3}}) = \frac{5}{6} \text{Cl}_2(\frac{\pi}{3}) - \frac{\pi \ln 3}{12}$.