

Cleo Bench Problem 6

Yet another log-sin integral $\int_0^{\pi/3} \log(1 + \sin x) \log(1 - \sin x) dx$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^{\pi/3} \log(1 + \sin x) \log(1 - \sin x) dx \approx -0.41142425522824105371 \dots$$

Result

With $G = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^2}$ (Catalan's constant), $\text{Cl}_2(\theta) = \sum_{n \geq 1} \frac{\sin n\theta}{n^2}$ (Clausen function), and the inverse tangent integral of order 3,

$$\text{Ti}_3(x) = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)^3} = \Im \text{Li}_3(ix),$$

the value is

$$I = -\frac{25\pi^3}{216} + \frac{\pi}{3} \log^2 2 - \log 2 \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi}{6} \log^2(2 + \sqrt{3}) + \frac{8G}{3} \log(2 + \sqrt{3}) + 4 \text{Ti}_3(2 - \sqrt{3})$$

Equivalently, using the inversion $\text{Ti}_3(2 - \sqrt{3}) = \frac{\pi}{4} \log^2(2 + \sqrt{3}) + \frac{\pi^3}{16} - \text{Ti}_3(2 + \sqrt{3})$,

$$I = \frac{29\pi^3}{216} + \frac{\pi}{3} \log^2 2 - \log 2 \text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{5\pi}{6} \log^2(2 + \sqrt{3}) + \frac{8G}{3} \log(2 + \sqrt{3}) - 4 \text{Ti}_3(2 + \sqrt{3}).$$

Numerically $I = -0.4114242552282410537103283722220855175392234960313920 \dots$

Throughout write $\lambda := \log(2 + \sqrt{3})$, so $\log(2 - \sqrt{3}) = -\lambda$ (since $(2 - \sqrt{3})(2 + \sqrt{3}) = 1$).

*Problem originally posed on Mathematics Stack Exchange ([question 1588996](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Derivation

0. Algebraic splitting

For $0 \leq x \leq \pi/3$ both $1 \pm \sin x > 0$, and $ab = \frac{(a+b)^2 - (a-b)^2}{4}$ gives

$$\log(1+\sin x) \log(1-\sin x) = \frac{1}{4} \log^2(1-\sin^2 x) - \frac{1}{4} \log^2 \frac{1+\sin x}{1-\sin x} = \log^2 \cos x - \log^2 \tan\left(\frac{\pi}{4} + \frac{x}{2}\right),$$

where we used $1 - \sin^2 x = \cos^2 x$ with $\cos x > 0$, and

$$\frac{1+\sin x}{1-\sin x} = \frac{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}{(\cos \frac{x}{2} - \sin \frac{x}{2})^2} = \tan^2\left(\frac{\pi}{4} + \frac{x}{2}\right) > 0.$$

In the second piece substitute $t = \frac{\pi}{4} + \frac{x}{2}$ ($dx = 2 dt$, $t: \frac{\pi}{4} \rightarrow \frac{5\pi}{12}$) and then $w = \tan t$ ($dt = \frac{dw}{1+w^2}$, $\tan \frac{5\pi}{12} = 2 + \sqrt{3}$):

$$\int_0^{\pi/3} \log^2 \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) dx = 2 \int_{\pi/4}^{5\pi/12} \log^2 \tan t dt = 2 \int_1^{2+\sqrt{3}} \frac{\log^2 w}{1+w^2} dw.$$

Hence

$$I = A - 2B, \quad A := \int_0^{\pi/3} \log^2 \cos x dx, \quad B := \int_1^{2+\sqrt{3}} \frac{\log^2 w}{1+w^2} dw. \quad (0.1)$$

1. The log-cosine part A

$$\mathbf{1.1 \text{ A Beta-function lemma:}} \quad \int_0^{\pi/2} \log^2(2 \cos x) dx = \frac{\pi^3}{24}$$

For $\Re s > -1$ the Beta integral and the Legendre duplication formula $\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2} + 1) = 2^{-s} \sqrt{\pi} \Gamma(s+1)$ give

$$\int_0^{\pi/2} (2 \cos x)^s dx = 2^s \cdot \frac{1}{2} B\left(\frac{s+1}{2}, \frac{1}{2}\right) = \frac{2^s \sqrt{\pi} \Gamma(\frac{s+1}{2})}{2 \Gamma(\frac{s}{2} + 1)} = \frac{\pi}{2} \frac{\Gamma(1+s)}{\Gamma(1+\frac{s}{2})^2} =: f(s).$$

Both sides are analytic in s near 0; differentiation under the integral sign is justified since $(2 \cos x)^s \log^k(2 \cos x)$ is dominated on $[0, \frac{\pi}{2}]$, uniformly for $|s| \leq \frac{1}{2}$, by the integrable function $((2 \cos x)^{1/2} + (2 \cos x)^{-1/2}) \log^k \frac{4}{2 \cos x} \cdot C$. From $\log \Gamma(1+u) = -\gamma u + \sum_{k \geq 2} \frac{(-1)^k \zeta(k)}{k} u^k$,

$$g(s) := \log \Gamma(1+s) - 2 \log \Gamma\left(1 + \frac{s}{2}\right) = \frac{\zeta(2)}{4} s^2 - \frac{\zeta(3)}{4} s^3 + O(s^4),$$

so $f(s) = \frac{\pi}{2} e^{g(s)}$ has $f''(0) = \frac{\pi}{2} g''(0) = \frac{\pi}{2} \cdot \frac{\pi^2}{12}$, i.e.

$$\int_0^{\pi/2} \log^2(2 \cos x) dx = f''(0) = \frac{\pi^3}{24}. \quad (1.1)$$

1.2 Complex representation

For $0 \leq x < \pi/2$ we have $1 + e^{2ix} = 2 \cos x e^{ix}$ with $\Re(1 + e^{2ix}) = 1 + \cos 2x > 0$, so for the principal logarithm

$$\log(1 + e^{2ix}) = \log(2 \cos x) + ix. \quad (1.2)$$

For $w = a + ib$ one has $a^2 = \Re(w^2) + b^2$; applying this to (1.2),

$$\log^2(2 \cos x) = \Re[\log^2(1 + e^{2ix})] + x^2. \quad (1.3)$$

1.3 An explicit antiderivative

Let

$$H(z) := \log^2(1-z) \log z + 2 \log(1-z) \operatorname{Li}_2(1-z) - 2 \operatorname{Li}_3(1-z),$$

with all branches principal. Using $\operatorname{Li}'_2(u) = -\frac{\log(1-u)}{u}$ and $\operatorname{Li}'_3(u) = \frac{\operatorname{Li}_2(u)}{u}$,

$$\begin{aligned} H'(z) &= \frac{\log^2(1-z)}{z} - \frac{2 \log(1-z) \log z}{1-z} + \left[\frac{2 \log(1-z) \log z}{1-z} - \frac{2 \operatorname{Li}_2(1-z)}{1-z} \right] + \frac{2 \operatorname{Li}_2(1-z)}{1-z} \\ &= \frac{\log^2(1-z)}{z}, \end{aligned} \tag{1.4}$$

valid wherever $z \notin (-\infty, 0]$ and $1-z \notin [1, \infty)$ (so that all principal branches are analytic; note $\frac{d}{du} \operatorname{Li}_2(u) = -\frac{\log(1-u)}{u}$ holds with the principal log whenever $u \notin [1, \infty)$).

Now parametrize $z(x) = -e^{2ix} = e^{i(2x-\pi)}$ for $x \in (\pi/3, \pi/2)$; then z runs over the arc $\{e^{i\phi} : \phi \in (-\pi/3, 0)\}$. Along it:

- $z \notin (-\infty, 0]$ and $\arg z \in (-\pi/3, 0)$, so $\log z$ is principal and analytic;
- $1-z = 1 - e^{i\phi} = 2 \sin \frac{|\phi|}{2} e^{i(\pi-|\phi|)/2}$ has modulus ≤ 1 and argument in $[\pi/3, \pi/2)$, hence stays off the cut $[1, \infty)$ (also at the endpoint $z_0 = e^{-i\pi/3}$, where $1-z_0 = e^{i\pi/3}$, the functions $\operatorname{Li}_2, \operatorname{Li}_3$ are analytic since $e^{i\pi/3} \notin [1, \infty)$).

Since $z'(x) = 2iz(x)$ and $1-z(x) = 1 + e^{2ix}$ (principal branch, by (1.2)),

$$\frac{d}{dx} H(z(x)) = H'(z) 2iz = 2i \log^2(1 + e^{2ix}).$$

As $x \rightarrow \pi/2^-$, $z \rightarrow 1$ along the circle and $H(z) \rightarrow 0$: indeed $\operatorname{Li}_3(1-z) \rightarrow 0$, $\log(1-z) \operatorname{Li}_2(1-z) = O(|1-z| \log |1-z|) \rightarrow 0$, and $\log^2(1-z) \log z = O(|1-z| \log^2 |1-z|) \rightarrow 0$ because $|\log z| = |2x - \pi| \asymp |1-z|$ on the circle. The integrand has only an integrable \log^2 singularity at $x = \pi/2$, so by the fundamental theorem of calculus (improper at the right endpoint)

$$\int_{\pi/3}^{\pi/2} \log^2(1 + e^{2ix}) dx = \frac{1}{2i} [0 - H(e^{-i\pi/3})] = \frac{i}{2} H(e^{-i\pi/3}). \tag{1.5}$$

1.4 Polylogarithms at $e^{i\pi/3}$

All series below converge absolutely on $|z| = 1$ for weight ≥ 2 , so rearrangement is legitimate, and $\operatorname{Li}_2, \operatorname{Li}_3$ are continuous on the closed unit disk (uniform convergence), so these series values agree with the principal branches used above.

(a) Multiplication formula. For $|z| \leq 1$ and $\omega = e^{2\pi i/3}$, summing the series,

$$\operatorname{Li}_s(z) + \operatorname{Li}_s(\omega z) + \operatorname{Li}_s(\omega^2 z) = 3^{1-s} \operatorname{Li}_s(z^3), \quad s = 2, 3.$$

Take $z = e^{i\pi/3}$; the three arguments are $e^{i\pi/3}, e^{i\pi} = -1, e^{i5\pi/3} = \overline{e^{i\pi/3}}$. Taking real parts and using $\Re \operatorname{Li}_s(\bar{u}) = \Re \operatorname{Li}_s(u)$, $\operatorname{Li}_2(-1) = -\frac{\pi^2}{12}$, $\operatorname{Li}_3(-1) = -\frac{3}{4}\zeta(3)$:

$$2 \Re \operatorname{Li}_2(e^{i\pi/3}) = \frac{1}{3} \left(-\frac{\pi^2}{12} \right) + \frac{\pi^2}{12} = \frac{\pi^2}{18} \implies \Re \operatorname{Li}_2(e^{i\pi/3}) = \frac{\pi^2}{36},$$

$$2 \Re \operatorname{Li}_3(e^{i\pi/3}) = \frac{1}{9} \left(-\frac{3}{4}\zeta(3) \right) + \frac{3}{4}\zeta(3) = \frac{2}{3}\zeta(3) \implies \Re \operatorname{Li}_3(e^{i\pi/3}) = \frac{\zeta(3)}{3}.$$

(b) The Glaisher sum Sl_3 . For $0 < \theta < 2\pi$ the series $\sum_{n \geq 1} \frac{e^{in\theta}}{n}$ converges (Dirichlet test), so by Abel's theorem it equals $-\log(1 - e^{i\theta})$; from $1 - e^{i\theta} = 2 \sin \frac{\theta}{2} e^{i(\theta-\pi)/2}$,

$$\sum_{n \geq 1} \frac{\sin n\theta}{n} = \frac{\pi - \theta}{2}.$$

To integrate this termwise, integrate the Abel means: for $0 < r < 1$, $\int_0^\theta \sum_n r^n \frac{\sin nt}{n} dt = \sum_n r^n \frac{1 - \cos n\theta}{n^2}$ (uniform convergence). As $r \rightarrow 1^-$ the left side tends to $\int_0^\theta \frac{\pi-t}{2} dt$ by dominated convergence, because $|\Im \log(1 - re^{it})| = |\arg(1 - re^{it})| \leq \frac{\pi}{2}$ uniformly (as $\Re(1 - re^{it}) > 0$); the right side tends to $\sum_n \frac{1 - \cos n\theta}{n^2}$ (dominated by $2/n^2$). Hence

$$\sum_{n \geq 1} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\pi\theta}{2} + \frac{\theta^2}{4}, \quad 0 \leq \theta \leq 2\pi, \quad (1.6)$$

and integrating once more (now the series converges uniformly),

$$\text{Sl}_3(\theta) := \sum_{n \geq 1} \frac{\sin n\theta}{n^3} = \frac{\pi^2\theta}{6} - \frac{\pi\theta^2}{4} + \frac{\theta^3}{12}, \quad 0 \leq \theta \leq 2\pi. \quad (1.7)$$

In particular $\Im \text{Li}_3(e^{i\pi/3}) = \text{Sl}_3(\pi/3) = \pi^3(\frac{1}{18} - \frac{1}{36} + \frac{1}{324}) = \frac{5\pi^3}{162}$. Also (1.6) at $\theta = \pi/3$ re-confirms (a). Therefore

$$\text{Li}_2(e^{i\pi/3}) = \frac{\pi^2}{36} + i \text{Cl}_2\left(\frac{\pi}{3}\right), \quad \text{Li}_3(e^{i\pi/3}) = \frac{\zeta(3)}{3} + \frac{5i\pi^3}{162}. \quad (1.8)$$

1.5 Assembling A

At $z_0 = e^{-i\pi/3}$: $\log z_0 = -\frac{i\pi}{3}$, $1 - z_0 = e^{i\pi/3}$, $\log(1 - z_0) = \frac{i\pi}{3}$, so by (1.8)

$$H(z_0) = \left(-\frac{\pi^2}{9}\right)\left(-\frac{i\pi}{3}\right) + \frac{2i\pi}{3}\left(\frac{\pi^2}{36} + i \text{Cl}_2\left(\frac{\pi}{3}\right)\right) - 2\left(\frac{\zeta(3)}{3} + \frac{5i\pi^3}{162}\right) = -\frac{2\pi}{3} \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{2\zeta(3)}{3} - \frac{i\pi^3}{162},$$

since $\frac{1}{27} + \frac{1}{54} - \frac{5}{81} = -\frac{1}{162}$. By (1.5),

$$\Re \int_{\pi/3}^{\pi/2} \log^2(1 + e^{2ix}) dx = \Re \left[\frac{i}{2} H(z_0) \right] = \frac{\pi^3}{324}.$$

Combining with (1.3),

$$\int_{\pi/3}^{\pi/2} \log^2(2 \cos x) dx = \frac{\pi^3}{324} + \int_{\pi/3}^{\pi/2} x^2 dx = \frac{\pi^3}{324} + \frac{\pi^3}{24} - \frac{\pi^3}{81} = \frac{7\pi^3}{216},$$

and with (1.1),

$$\int_0^{\pi/3} \log^2(2 \cos x) dx = \frac{\pi^3}{24} - \frac{7\pi^3}{216} = \frac{\pi^3}{108}. \quad (1.9)$$

Next, the weight-2 analogue. With the same parametrization on $x \in (0, \pi/3)$, i.e. the arc $\{e^{i\phi} : \phi \in (-\pi, -\pi/3)\}$, we have $-\text{Li}'_2(z) = \frac{\log(1-z)}{z}$ with all branches principal ($1 - z$ lies in the closed upper half plane off $[1, \infty)$, z off $[0, \infty)$); both endpoints are regular points of Li_2), so

$$\int_0^{\pi/3} \log(1 + e^{2ix}) dx = \frac{1}{2i} \left[\text{Li}_2(-1) - \text{Li}_2(e^{-i\pi/3}) \right] = \frac{1}{2i} \left[-\frac{\pi^2}{12} - \frac{\pi^2}{36} + i \text{Cl}_2\left(\frac{\pi}{3}\right) \right],$$

using $\text{Li}_2(e^{-i\pi/3}) = \overline{\text{Li}_2(e^{i\pi/3})}$. Taking real parts and using (1.2):

$$\int_0^{\pi/3} \log(2 \cos x) dx = \frac{1}{2} \text{Cl}_2\left(\frac{\pi}{3}\right). \quad (1.10)$$

Finally, from $\log \cos x = \log(2 \cos x) - \log 2$ and (1.9)–(1.10):

$$\begin{aligned} A &= \int_0^{\pi/3} \log^2(2 \cos x) dx - 2 \log 2 \int_0^{\pi/3} \log(2 \cos x) dx + \frac{\pi}{3} \log^2 2 \\ &= \frac{\pi^3}{108} - \log 2 \operatorname{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi}{3} \log^2 2. \end{aligned} \quad (1.11)$$

2. The log-tangent part B

Define for $x > 0$

$$\operatorname{Ti}_2(x) = \int_0^x \frac{\arctan t}{t} dt, \quad \operatorname{Ti}_3(x) = \int_0^x \frac{\operatorname{Ti}_2(t)}{t} dt;$$

for $|x| \leq 1$ these equal the series $\sum_k \frac{(-1)^k x^{2k+1}}{(2k+1)^2}$, $\sum_k \frac{(-1)^k x^{2k+1}}{(2k+1)^3}$. Moreover $\operatorname{Ti}_k(x) = \Im \operatorname{Li}_k(ix)$ for all $x > 0$: the imaginary axis avoids the cut $[1, \infty)$, and $\frac{d}{dx} \Im \operatorname{Li}_2(ix) = \Im \frac{-\log(1-ix)}{x} = \frac{\arctan x}{x}$, $\frac{d}{dx} \Im \operatorname{Li}_3(ix) = \frac{\Im \operatorname{Li}_2(ix)}{x}$, with all functions vanishing at $x = 0$.

2.1 Antiderivative

$$\begin{aligned} \frac{d}{dw} \left[\log^2 w \arctan w - 2 \log w \operatorname{Ti}_2(w) + 2 \operatorname{Ti}_3(w) \right] \\ &= \frac{\log^2 w}{1+w^2} + \frac{2 \log w \arctan w}{w} - \frac{2 \operatorname{Ti}_2(w)}{w} - \frac{2 \log w \arctan w}{w} + \frac{2 \operatorname{Ti}_2(w)}{w} \\ &= \frac{\log^2 w}{1+w^2}. \end{aligned}$$

Hence, with $c := 2 + \sqrt{3}$, $\arctan c = \frac{5\pi}{12}$, $\log c = \lambda$:

$$B = \left[\log^2 w \arctan w - 2 \log w \operatorname{Ti}_2(w) + 2 \operatorname{Ti}_3(w) \right]_1^c = \frac{5\pi}{12} \lambda^2 - 2\lambda \operatorname{Ti}_2(c) + 2 \operatorname{Ti}_3(c) - 2 \operatorname{Ti}_3(1). \quad (2.1)$$

2.2 $\operatorname{Ti}_3(1) = \frac{\pi^3}{32}$

By (1.7): $\operatorname{Ti}_3(1) = \sum_k \frac{(-1)^k}{(2k+1)^3} = \sum_n \frac{\sin(n\pi/2)}{n^3} = \operatorname{Sl}_3(\pi/2) = \pi^3 \left(\frac{1}{12} - \frac{1}{16} + \frac{1}{96} \right) = \frac{\pi^3}{32}$.

2.3 Ti_2 at $2 \pm \sqrt{3}$

(a) A Clausen identity. For $0 < \sigma \leq \pi$, parametrizing the upper unit arc $z = e^{i\phi}$, $\phi \in (0, \sigma]$, and using $-\operatorname{Li}'_2(z) = \frac{\log(1-z)}{z}$ (principal branches; $1 - e^{i\phi} = 2 \sin \frac{\phi}{2} e^{i(\phi-\pi)/2}$ stays off both cuts, and Li_2 is continuous up to $z = 1$):

$$\int_0^\sigma \log(1 - e^{i\phi}) d\phi = \frac{1}{i} \left[\operatorname{Li}_2(1) - \operatorname{Li}_2(e^{i\sigma}) \right],$$

the improper endpoint $\phi = 0$ being harmless (logarithmic singularity). Real parts give

$$\int_0^\sigma \log \left(2 \sin \frac{\phi}{2} \right) d\phi = -\operatorname{Cl}_2(\sigma) \implies \int_0^\theta \log(2 \sin t) dt = -\frac{1}{2} \operatorname{Cl}_2(2\theta) \quad (0 < \theta \leq \frac{\pi}{2}),$$

and, by $t \mapsto \frac{\pi}{2} - t$ and $\operatorname{Cl}_2(\pi) = 0$,

$$\int_0^\theta \log(2 \cos t) dt = \frac{1}{2} \operatorname{Cl}_2(\pi - 2\theta).$$

Hence

$$\int_0^{\pi/12} \log \tan t \, dt = -\frac{1}{2} \left[\text{Cl}_2\left(\frac{\pi}{6}\right) + \text{Cl}_2\left(\frac{5\pi}{6}\right) \right]. \quad (2.2)$$

(b) Character evaluation. Since $\sin \frac{5n\pi}{6} = \sin(n\pi - \frac{n\pi}{6}) = -(-1)^n \sin \frac{n\pi}{6}$, even n cancel and odd n double in (2.2):

$$\text{Cl}_2\left(\frac{\pi}{6}\right) + \text{Cl}_2\left(\frac{5\pi}{6}\right) = 2 \sum_{m \text{ odd}} \frac{\sin(m\pi/6)}{m^2}.$$

For odd m : if $3 \nmid m$ then $\sin \frac{m\pi}{6} = \frac{1}{2}\chi_{-4}(m)$ and if $m = 3m'$ then $\sin \frac{m\pi}{6} = \sin \frac{m'\pi}{2} = \chi_{-4}(m')$, where $\chi_{-4}(m) = (-1)^{(m-1)/2}$ (check the six odd residues mod 12). With $\sum_m \frac{\chi_{-4}(m)}{m^2} = G$ and multiplicativity ($\chi_{-4}(3) = -1$), all series being absolutely convergent:

$$\sum_{m \text{ odd}} \frac{\sin(m\pi/6)}{m^2} = \frac{1}{2} \left(G - \frac{\chi_{-4}(3)}{9} G \right) + \frac{1}{9} G = \frac{1}{2} \cdot \frac{10G}{9} + \frac{G}{9} = \frac{2G}{3}.$$

Hence

$$\int_0^{\pi/12} \log \tan t \, dt = -\frac{2G}{3}. \quad (2.3)$$

(c) From the integral to Ti_2 . Integrating by parts ($t \log \tan t \rightarrow 0$ as $t \rightarrow 0^+$) and substituting $w = \tan t$ (so $\frac{2t \, dt}{\sin 2t} = \frac{\arctan w}{w} dw$):

$$\int_0^\theta \log \tan t \, dt = \theta \log \tan \theta - \int_0^\theta \frac{2t}{\sin 2t} dt = \theta \log \tan \theta - \text{Ti}_2(\tan \theta).$$

At $\theta = \frac{\pi}{12}$, $\tan \frac{\pi}{12} = 2 - \sqrt{3}$, so with (2.3):

$$\text{Ti}_2(2 - \sqrt{3}) = \frac{2G}{3} - \frac{\pi}{12} \lambda. \quad (2.4)$$

For $x > 0$, $\frac{d}{dx} [\text{Ti}_2(x) - \text{Ti}_2(1/x)] = \frac{\arctan x + \arctan(1/x)}{x} = \frac{\pi}{2x}$ and the bracket vanishes at $x = 1$, whence $\text{Ti}_2(x) - \text{Ti}_2(1/x) = \frac{\pi}{2} \log x$ and

$$\text{Ti}_2(2 + \sqrt{3}) = \frac{2G}{3} + \frac{5\pi}{12} \lambda. \quad (2.5)$$

2.4 Inversion for Ti_3

For the principal branch, $\text{Li}_2(z) + \text{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-z)$ on $\mathbb{C} \setminus [0, \infty)$: both sides are analytic there (the cuts of $\text{Li}_k(z)$, $\text{Li}_k(1/z)$, $\log(-z)$ all lie in $[0, \infty)$), their derivatives agree because $\log(1 - 1/z) - \log(1 - z) = \log(-1/z)$ holds with principal branches on this domain (both sides are analytic, differ by a constant in $2\pi i\mathbb{Z}$, and agree at $z = -1$), and at $z = -1$ both sides equal $-\frac{\pi^2}{6}$. Consequently, for $F(z) := \text{Li}_3(z) - \text{Li}_3(1/z)$ one gets $F'(z) = \frac{1}{z} [\text{Li}_2(z) + \text{Li}_2(1/z)]$ and, integrating from $z = -1$ (where $F = 0$, $\log(-z) = 0$),

$$\text{Li}_3(z) - \text{Li}_3(1/z) = -\frac{\pi^2}{6} \log(-z) - \frac{1}{6} \log^3(-z), \quad z \in \mathbb{C} \setminus [0, \infty).$$

Put $z = ix$, $x > 0$ (inside the domain): $\log(-ix) = \log x - \frac{i\pi}{2}$, $\Im \text{Li}_3(ix) = \text{Ti}_3(x)$, $\Im \text{Li}_3(-i/x) = -\text{Ti}_3(1/x)$ (conjugation symmetry). Taking imaginary parts, with $\Im(\log x - \frac{i\pi}{2})^3 = -\frac{3\pi}{2} \log^2 x + \frac{\pi^3}{8}$:

$$\text{Ti}_3(x) + \text{Ti}_3\left(\frac{1}{x}\right) = \frac{\pi}{4} \log^2 x + \frac{\pi^3}{16}, \quad x > 0 \quad (2.6)$$

(consistency check at $x = 1$: $2 \cdot \frac{\pi^3}{32} = \frac{\pi^3}{16}$ ✓). In particular

$$\text{Ti}_3(2 + \sqrt{3}) = \frac{\pi}{4} \lambda^2 + \frac{\pi^3}{16} - \text{Ti}_3(2 - \sqrt{3}). \quad (2.7)$$

2.5 Value of B

Substituting (2.5) and $\text{Ti}_3(1) = \frac{\pi^3}{32}$ into (2.1):

$$B = \frac{5\pi}{12}\lambda^2 - 2\lambda\left(\frac{2G}{3} + \frac{5\pi}{12}\lambda\right) + 2\text{Ti}_3(c) - \frac{\pi^3}{16} = -\frac{5\pi}{12}\lambda^2 - \frac{4G}{3}\lambda + 2\text{Ti}_3(2 + \sqrt{3}) - \frac{\pi^3}{16}. \quad (2.8)$$

3. Assembly

By (0.1), (1.11), (2.8):

$$I = A - 2B = \frac{\pi^3}{108} - \log 2 \text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi}{3} \log^2 2 + \frac{5\pi}{6}\lambda^2 + \frac{8G}{3}\lambda - 4\text{Ti}_3(2 + \sqrt{3}) + \frac{\pi^3}{8},$$

and since $\frac{1}{108} + \frac{1}{8} = \frac{29}{216}$,

$$I = \frac{29\pi^3}{216} + \frac{\pi}{3} \log^2 2 - \log 2 \text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{5\pi}{6}\lambda^2 + \frac{8G}{3}\lambda - 4\text{Ti}_3(2 + \sqrt{3}).$$

Using the inversion (2.7), $\frac{29}{216} - \frac{1}{4} = -\frac{25}{216}$ and $\frac{5\pi}{6} - \pi = -\frac{\pi}{6}$:

$$I = -\frac{25\pi^3}{216} + \frac{\pi}{3} \log^2 2 - \log 2 \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi}{6} \log^2(2 + \sqrt{3}) + \frac{8G}{3} \log(2 + \sqrt{3}) + 4\text{Ti}_3(2 - \sqrt{3}).$$

■

Numerical verification

All computations with `mpmath` (scripts in `results/scratch/work/q1588996-.../`), $\text{Ti}_3(x)$ computed as `\Im Li3(ix)`, Cl_2 as `clsin(2,.)`.

- Direct quadrature at `mp.dps = 110` of $\int_0^{\pi/3} \log(1 + \sin x) \log(1 - \sin x) dx$ (smooth integrand, split at $\pi/6$):

$$I = -0.4114242552282410537103283722220855175392234960313920 \\ 1139076283561929055384357879120607166\dots$$

- Closed form at the same precision:

$$F = -0.4114242552282410537103283722220855175392234960313920 \\ 1139076283561929055384357879120607166\dots$$

- $|I - F| \approx 4.2 \cdot 10^{-111}$: **109 significant digits agree** (also matches the 20 digits quoted in the problem statement).

Every intermediate identity was verified independently to 35–60 digits: the split $I = A - 2B$; (1.1) $\int_0^{\pi/2} \log^2(2 \cos) = \pi^3/24$; (1.9) $\int_0^{\pi/3} \log^2(2 \cos) = \pi^3/108$; (1.10) $\int_0^{\pi/3} \log(2 \cos) = \frac{1}{2} \text{Cl}_2(\pi/3)$; the contour identity (1.5) (complex-valued, both sides); $H(e^{-i\pi/3})$; (2.1); (2.5); (2.6); and the final formula in both equivalent forms.

A PSLQ search (120–220 digits) for a rational linear relation between $\text{Ti}_3(2 - \sqrt{3})$ and the natural weight-3 basis $\{\zeta(3), \pi^3, \pi G, G \log 2, G \log 3, G\lambda, \text{Cl}_2(\pi/3) \cdot \{\pi, \log 2, \log 3, \lambda\}, \pi^2 \cdot \{\log 2, \log 3, \lambda\}, \pi \cdot \{\log^2 2, \log^2 3, \lambda^2, \lambda \log 2, \lambda \log 3, \log 2 \log 3\}\}$ found none (candidate relations with coefficients $\sim 10^{10}$ failed at 220-digit precision), so no further reduction of the answer is expected: $\text{Ti}_3(2 - \sqrt{3}) = \Im \text{Li}_3(i(2 - \sqrt{3})) \approx 0.2672979725$ appears to be an irreducible constant of this problem.

Notes

- The derivation is complete and self-contained: every branch choice is specified (principal branches throughout, with paths shown to avoid all cuts), every termwise integration is justified (Abel means + dominated convergence, or absolute/uniform convergence), and differentiation under the integral sign in §1.1 is dominated explicitly.
- Special values used and proved here: $\text{Li}_2(e^{i\pi/3})$, $\text{Li}_3(e^{i\pi/3})$ (multiplication formula + Bernoulli–Fourier sums), $\text{Ti}_2(2 \pm \sqrt{3})$ (classical Lewin values, re-derived via (2.2)–(2.4)), $\text{Ti}_3(1) = \beta(3) = \pi^3/32$, and the inversion formulas for Ti_2 , Ti_3 . Standard inputs assumed without re-proof: $\text{Li}_2(-1) = -\pi^2/12$, $\text{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ (i.e. $\eta(2), \eta(3)$), $L(\chi_{-4}, 2) = G$ (definition of Catalan’s constant), the Beta integral and Legendre duplication.
- The only non-elementary constants in the answer, $\text{Cl}_2(\pi/3)$ (Gieseking’s constant), G , and $\text{Ti}_3(2 - \sqrt{3})$, are standard “level 12” constants; the PSLQ experiment above supports that $\text{Ti}_3(2 - \sqrt{3})$ cannot be eliminated.