Cleo Bench Problem 20

Closed form solution to $\int_0^1 \arctan^2(x)\,\sqrt{x}\,dx$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^1 \arctan^2(x)\,\sqrt{x}\;dx.$$

Result

$$\boxed{I = \frac{\pi^2}{24} + \frac{\sqrt{2}\,\pi^2}{12} + \frac{2(\sqrt{2}-1)\pi}{3} + \frac{\sqrt{2}\,\pi}{6}\ln\frac{1+\sqrt{2}}{2} - \frac{4\sqrt{2}}{3}\ln\Big(1+\sqrt{2}\Big) - \frac{2\sqrt{2}}{3}\,\chi_2\Big(\frac{1}{\sqrt{2}}\Big)}$$

where χ_2 is the Legendre chi function,

$$\chi_2(z) = \sum_{n\ge 0}\frac{z^{2n+1}}{(2n+1)^2} = \tfrac{1}{2}\big[\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\big], \qquad \chi_2\Big(\tfrac{1}{\sqrt{2}}\Big) = \frac{1}{\sqrt{2}}\sum_{n\ge 0}\frac{1}{2^n(2n+1)^2} = 0.75606449\ldots$$

Numerically $I = 0.20649083085518026919342250406354 69\ldots$

An equivalent form (via a Landen identity, proved in the Notes) is

$$I = \frac{\pi^2}{24} + \frac{2(\sqrt{2}-1)\pi}{3} + \frac{\sqrt{2}\,\pi}{6}\ln\frac{1+\sqrt{2}}{2} + \frac{\sqrt{2}}{3}\ln 2\,\ln(1+\sqrt{2}) - \frac{4\sqrt{2}}{3}\ln(1+\sqrt{2}) + \frac{2\sqrt{2}}{3}\,\chi_2(3-2\sqrt{2}).$$

Derivation

Throughout, $L := \ln(1+\sqrt{2})$, and all arctangents take principal values.

Step 0. Two elementary antiderivatives

For all real t one has $t^4+1 = (t^2-\sqrt{2}\,t+1)(t^2+\sqrt{2}\,t+1)$, and both quadratic factors are positive ($t^2 \pm \sqrt{2}t + 1 = (t \pm \frac{\sqrt{2}}{2})^2 + \frac{1}{2}$). Direct differentiation verifies, on $t > 0$:

$$\frac{d}{dt}\left[\frac{1}{\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}\,t}\right] = \frac{1}{\sqrt{2}}\cdot\frac{\frac{1}{\sqrt{2}}\big(1+\frac{1}{t^2}\big)}{1+\frac{(t^2-1)^2}{2t^2}} = \frac{t^2+1}{t^4+1},$$

$$\frac{d}{dt}\left[\frac{1}{2\sqrt{2}}\ln\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1}\right] = \frac{1}{2\sqrt{2}}\cdot\frac{(2t-\sqrt{2})(t^2+\sqrt{2}t+1)-(2t+\sqrt{2})(t^2-\sqrt{2}t+1)}{t^4+1} = \frac{t^2-1}{t^4+1},$$

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[*] Problem originally posed on Mathematics Stack Exchange (question 1550806, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

the last equality because the numerator equals $(2t^3+\sqrt{2}t^2-\sqrt{2})-(2t^3-\sqrt{2}t^2+\sqrt{2}) = 2\sqrt{2}\,(t^2-1)$.

Consequently, using $\arctan\frac{t^2-1}{\sqrt{2}t} \to -\frac{\pi}{2}$ as $t \to 0^+$ and $\to +\frac{\pi}{2}$ as $t\to\infty$, and $\frac{2-\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2$:

$$\int_0^1 \frac{t^2+1}{t^4+1}\,dt = \frac{\pi}{2\sqrt{2}}, \qquad \int_0^1 \frac{t^2-1}{t^4+1}\,dt = \frac{1}{2\sqrt{2}}\ln(\sqrt{2}-1)^2 = -\frac{L}{\sqrt{2}},$$

$$\int_0^\infty \frac{t^2+1}{t^4+1}\,dt = \frac{\pi}{\sqrt{2}}, \qquad \int_0^\infty \frac{t^2-1}{t^4+1}\,dt = 0.$$

Adding/subtracting and halving:

$$F_2(1) := \int_0^1 \frac{t^2}{t^4+1}\,dt = \frac{\pi}{4\sqrt{2}} - \frac{L}{2\sqrt{2}}, \qquad \int_0^\infty \frac{t^2}{t^4+1}\,dt = \int_0^\infty \frac{dt}{t^4+1} = \frac{\pi}{2\sqrt{2}}. \tag{0}$$

Also $\displaystyle\int_0^1 \frac{2t\,dt}{1+t^4} = \arctan(t^2)\Big|_0^1 = \frac{\pi}{4}$.

Step 1. Integration by parts

With $u=\arctan^2 x$, $dv = x^{1/2}dx$, $v=\frac{2}{3}x^{3/2}$ (all terms continuous on $[0,1]$):

$$I = \frac{2}{3}\arctan^2(1) - \frac{4}{3}\int_0^1 \frac{x^{3/2}\arctan x}{1+x^2}\,dx = \frac{\pi^2}{24} - \frac{4}{3}J, \qquad J := \int_0^1 \frac{x^{3/2}\arctan x}{1+x^2}\,dx. \tag{1}$$

Step 2. Splitting J

Since $\dfrac{x^{3/2}}{1+x^2} = x^{-1/2} - \dfrac{x^{-1/2}}{1+x^2}$, and both resulting integrals converge absolutely ($\arctan x/\sqrt{x} \sim \sqrt{x}$ at 0),

$$J = J_1 - J_2, \qquad J_1 = \int_0^1 \frac{\arctan x}{\sqrt{x}}\,dx, \qquad J_2 = \int_0^1 \frac{\arctan x}{\sqrt{x}\,(1+x^2)}\,dx.$$

The substitution $x=t^2$ ($dx = 2t\,dt$) gives

$$J_1 = 2\int_0^1 \arctan(t^2)\,dt, \qquad J_2 = 2\int_0^1 \frac{\arctan(t^2)}{1+t^4}\,dt =: 2A. \tag{2}$$

Step 3. Evaluation of J_1

Integrating by parts and using (0):

$$\int_0^1 \arctan(t^2)\,dt = \Big[t\arctan(t^2)\Big]_0^1 - \int_0^1 \frac{2t^2}{1+t^4}\,dt = \frac{\pi}{4} - 2\left(\frac{\pi}{4\sqrt{2}} - \frac{L}{2\sqrt{2}}\right),$$

hence

$$J_1 = \frac{\pi}{2} - \frac{\pi}{\sqrt{2}} + \sqrt{2}\,L. \tag{3}$$

Step 4. The combination $A - C$ via inversion

Set

$$A = \int_0^1 \frac{\arctan(x^2)}{1+x^4}\,dx, \qquad C = \int_0^1 \frac{x^2\arctan(x^2)}{1+x^4}\,dx, \qquad K_\infty = \int_0^\infty \frac{\arctan(x^2)}{1+x^4}\,dx.$$

(4a) K_∞ in closed form. For every $x > 0$, $\arctan(x^2) = \int_0^1 \frac{x^2}{1+s^2x^4}\,ds$ (the s-antiderivative of the integrand is $\arctan(sx^2)$). Therefore

$$K_\infty = \int_0^\infty \int_0^1 \frac{x^2}{(1+x^4)(1+s^2x^4)}\,ds\,dx.$$

The integrand is nonnegative and measurable, so Tonelli's theorem allows swapping the order. For fixed $s \in (0,1)$, the algebraic identity

$$\frac{1}{(1+x^4)(1+s^2x^4)} = \frac{1}{1-s^2}\left[\frac{1}{1+x^4} - \frac{s^2}{1+s^2x^4}\right]$$

(check by putting the right side over a common denominator: the numerator is $(1+s^2x^4) - s^2(1+x^4) = 1-s^2$) together with (0) and the scaling $u = \sqrt{s}\,x$, which gives $\int_0^\infty \frac{s^2x^2}{1+s^2x^4}dx = \sqrt{s}\,\frac{\pi}{2\sqrt{2}}$, yields

$$\int_0^\infty \frac{x^2\,dx}{(1+x^4)(1+s^2x^4)} = \frac{\pi}{2\sqrt{2}} \cdot \frac{1-\sqrt{s}}{1-s^2} = \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{(1+\sqrt{s})(1+s)},$$

using $1 - s^2 = (1-\sqrt{s})(1+\sqrt{s})(1+s)$. With $s = w^2$ and the partial fraction $\frac{w}{(1+w)(1+w^2)} = \frac{1}{2}\left[\frac{w+1}{1+w^2} - \frac{1}{1+w}\right]$,

$$\int_0^1 \frac{ds}{(1+\sqrt{s})(1+s)} = 2\int_0^1 \frac{w\,dw}{(1+w)(1+w^2)} = \left[\tfrac{1}{2}\ln(1+w^2) + \arctan w - \ln(1+w)\right]_0^1 = \frac{1}{2}\ln 2 + \frac{\pi}{4} - \ln 2 = \frac{\pi}{4} -$$

Hence

$$K_\infty = \frac{\pi^2}{8\sqrt{2}} - \frac{\pi\ln 2}{4\sqrt{2}}. \tag{4a}$$

(4b) Inversion. In $\int_1^\infty \frac{\arctan(x^2)}{1+x^4}dx$ substitute $x = 1/u$ and use $\arctan(u^{-2}) = \frac{\pi}{2} - \arctan(u^2)$ for $u > 0$:

$$\int_1^\infty \frac{\arctan(x^2)}{1+x^4}\,dx = \int_0^1 \frac{u^2\left[\frac{\pi}{2} - \arctan(u^2)\right]}{1+u^4}\,du = \frac{\pi}{2}F_2(1) - C.$$

Thus $K_\infty = A + \frac{\pi}{2}F_2(1) - C$, and with (0) and $(4a)$,

$$A - C = K_\infty - \frac{\pi}{2}\Big(\frac{\pi}{4\sqrt{2}} - \frac{L}{2\sqrt{2}}\Big) = \frac{\pi}{4\sqrt{2}}\,(L - \ln 2) = \frac{\pi}{4\sqrt{2}}\ln\frac{1+\sqrt{2}}{2}. \tag{4}$$

Step 5. The combination $A + C$ via parts

$$A + C = \int_0^1 \frac{(1+x^2)\arctan(x^2)}{1+x^4}\,dx.$$

Define, for $x \in [0,1)$,

$$v(x) = \frac{1}{\sqrt{2}}\left[\arctan\frac{\sqrt{2}\,x}{1-x^2} - \frac{\pi}{2}\right], \qquad v(1) := 0.$$

With $g(x)=\frac{\sqrt{2}x}{1-x^2}$ one computes $g'=\frac{\sqrt{2}(1+x^2)}{(1-x^2)^2}$ and $1+g^2=\frac{(1-x^2)^2+2x^2}{(1-x^2)^2}=\frac{1+x^4}{(1-x^2)^2}$, so on $[0,1)$

$$v'(x)=\frac{1+x^2}{1+x^4},$$

and v is continuous on $[0,1]$ because $g(x)\to+\infty$, hence $\arctan g\to\frac{\pi}{2}$, as $x\to1^-$. Integration by parts (the boundary terms vanish: $\arctan(0^2)=0$ at $x=0$ and $v(1)=0$ at $x=1$) gives

$$A+C=-\int_0^1 v(x)\,\frac{2x}{1+x^4}\,dx=\frac{\pi}{2\sqrt{2}}\int_0^1\frac{2x\,dx}{1+x^4}-\sqrt{2}\,D=\frac{\pi^2}{8\sqrt{2}}-\sqrt{2}\,D, \tag{5}$$

where (using $\int_0^1\frac{2x}{1+x^4}dx=\frac{\pi}{4}$ from Step 0)

$$D:=\int_0^1\frac{x}{1+x^4}\,\arctan\frac{\sqrt{2}\,x}{1-x^2}\,dx.$$

Step 6. Evaluation of D: reduction to $\chi_2\left(\frac{1}{\sqrt{2}}\right)$

(6a) An exact trigonometric identity. Let $\theta=\arctan(x^2)\in[0,\frac{\pi}{4}]$ and $\phi(x)=\arctan\frac{\sqrt{2}x}{1-x^2}\in[0,\frac{\pi}{2}]$ (with $\phi(1)=\frac{\pi}{2}$ by continuity), for $x\in[0,1]$. Then

$$\sin^2\phi=\frac{\tan^2\phi}{1+\tan^2\phi}=\frac{2x^2}{(1-x^2)^2+2x^2}=\frac{2x^2}{1+x^4},\qquad \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2x^2}{1+x^4}.$$

Hence $\sin^2\phi=\sin2\theta$, and since $\phi\in[0,\frac{\pi}{2}]$,

$$\phi=\arcsin\sqrt{\sin2\theta}\,.$$

(6b) Change of variables. The map $x\mapsto\theta=\arctan(x^2)$ is a C^1 increasing bijection $[0,1]\to[0,\frac{\pi}{4}]$ with $d\theta=\frac{2x}{1+x^4}dx$. Therefore

$$D=\frac{1}{2}\int_0^{\pi/4}\arcsin\sqrt{\sin2\theta}\;d\theta=\frac{1}{4}\int_0^{\pi/2}\arcsin\sqrt{\sin u}\;du\qquad(u=2\theta).$$

Next substitute $u=\arcsin(t^2)$, $t\in[0,1]$, an increasing bijection with $du=\frac{2t}{\sqrt{1-t^4}}dt$ (the endpoint singularity at $t=1$ is of type $(1-t)^{-1/2}$, absolutely integrable):

$$D=\frac{1}{2}\int_0^1\frac{t\,\arcsin t}{\sqrt{1-t^4}}\,dt.$$

Now put $t=\sin w$, $w\in[0,\frac{\pi}{2}]$; since $\sqrt{1-t^4}=\cos w\,\sqrt{1+\sin^2 w}$ and $1+\sin^2 w=2-\cos^2 w$,

$$D=\frac{1}{2}\int_0^{\pi/2}\frac{w\,\sin w}{\sqrt{2-\cos^2 w}}\,dw,$$

a proper integral of a continuous function.

(6c) Integration by parts. Because

$$\frac{d}{dw}\left[-\arcsin\frac{\cos w}{\sqrt{2}}\right]=\frac{\sin w/\sqrt{2}}{\sqrt{1-\cos^2 w/2}}=\frac{\sin w}{\sqrt{2-\cos^2 w}},$$

we get, with vanishing boundary terms ($w=0$ kills the first factor; $\arcsin0=0$ at $w=\frac{\pi}{2}$),

$$D=\frac{1}{2}\left\{\left[-w\arcsin\tfrac{\cos w}{\sqrt{2}}\right]_0^{\pi/2}+\int_0^{\pi/2}\arcsin\frac{\cos w}{\sqrt{2}}\,dw\right\}=\frac{1}{2}\int_0^{\pi/2}\arcsin\!\left(\frac{\sin w}{\sqrt{2}}\right)dw,$$

the last step by $w \mapsto \frac{\pi}{2} - w$.

(6d) A classical parameter integral. For $k \in [0, \frac{1}{\sqrt{2}}]$ define $g(k) = \int_0^{\pi/2} \arcsin(k \sin w)\, dw$. The partial derivative $\frac{\sin w}{\sqrt{1-k^2\sin^2 w}}$ is continuous on $[0, \frac{1}{\sqrt{2}}] \times [0, \frac{\pi}{2}]$ and bounded by $\sqrt{2}$, so differentiation under the integral sign is justified (dominated convergence applied to difference quotients). With $c = \cos w$:

$$g'(k) = \int_0^{\pi/2} \frac{\sin w\, dw}{\sqrt{1-k^2\sin^2 w}} = \int_0^1 \frac{dc}{\sqrt{(1-k^2)+k^2c^2}} = \frac{1}{k} \operatorname{arcsinh} \frac{k}{\sqrt{1-k^2}} = \frac{1}{k}\cdot\frac{1}{2} \ln \frac{1+k}{1-k} = \frac{\operatorname{artanh} k}{k},$$

where $\operatorname{arcsinh} \frac{k}{\sqrt{1-k^2}} = \ln \frac{k+1}{\sqrt{1-k^2}} = \frac{1}{2} \ln \frac{1+k}{1-k}$. Since $g(0) = 0$ and $\frac{\operatorname{artanh} k}{k} = \sum_{n\ge 0} \frac{k^{2n}}{2n+1}$ converges uniformly on $[0, \frac{1}{\sqrt{2}}]$, termwise integration gives

$$g(k) = \int_0^k \frac{\operatorname{artanh} t}{t}\, dt = \sum_{n\ge 0} \frac{k^{2n+1}}{(2n+1)^2} = \chi_2(k).$$

Hence

$$D = \frac{1}{2}\, \chi_2\Big(\frac{1}{\sqrt{2}}\Big). \tag{6}$$

Step 7. Assembling everything

From (4), (5), (6):

$$A = \frac{(A+C)+(A-C)}{2} = \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{8\sqrt{2}} \ln \frac{1+\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\, \chi_2\Big(\frac{1}{\sqrt{2}}\Big). \tag{7}$$

From (1)–(3) and (7), $I = \frac{\pi^2}{24} - \frac{4}{3} J_1 + \frac{8}{3} A$, i.e.

$$I = \frac{\pi^2}{24} - \frac{4}{3}\left(\frac{\pi}{2} - \frac{\pi}{\sqrt{2}} + \sqrt{2} L\right) + \frac{8}{3}\left(\frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{8\sqrt{2}} \ln \frac{1+\sqrt{2}}{2} - \frac{\sqrt{2}}{4} \chi_2\big(\tfrac{1}{\sqrt{2}}\big)\right).$$

Rationalizing ($\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$):

$$\boxed{I = \frac{\pi^2}{24} + \frac{\sqrt{2}\,\pi^2}{12} + \frac{2(\sqrt{2}-1)\pi}{3} + \frac{\sqrt{2}\,\pi}{6} \ln \frac{1+\sqrt{2}}{2} - \frac{4\sqrt{2}}{3} \ln(1+\sqrt{2}) - \frac{2\sqrt{2}}{3}\, \chi_2\Big(\frac{1}{\sqrt{2}}\Big)}$$

with $\chi_2\big(\frac{1}{\sqrt{2}}\big) = \frac{1}{2}\left[\mathrm{Li}_2\big(\frac{\sqrt{2}}{2}\big) - \mathrm{Li}_2\big(-\frac{\sqrt{2}}{2}\big)\right]$.

Numerical verification

All computations with mpmath at `mp.dps = 60` and re-run at `mp.dps = 120` (script: `results/scratch/work/`

Every intermediate identity was checked independently:

check	quantity	residual
1	$I = \frac{\pi^2}{24} - \frac{4}{3} J$	$6 \cdot 10^{-62}$
2	$J = J_1 - 2A$	0
3	J_1 closed form (3)	0
4	$A - C$ formula (4)	$2 \cdot 10^{-62}$
4b	K_∞ formula ($4a$)	$1 \cdot 10^{-61}$
5	$A + C = \frac{\pi^2}{8\sqrt{2}} - \sqrt{2} D$	$2 \cdot 10^{-61}$
6a–6d	each link of the D-chain, and $D = \frac{1}{2}\chi_2(\frac{1}{\sqrt{2}})$	$\le 5 \cdot 10^{-33}$ (one midpoint-singular quad), others ≤ 4

Final comparison at 120 digits:

- direct quadrature: $I = 0.2064908308551802691934225040635469051180165142744 3\ldots$
- closed form: $I_{\mathrm{cf}} = 0.2064908308551802691934225040635469051180165142744 3\ldots$
- $|I_{\mathrm{cf}} - I| \approx 4.8 \cdot 10^{-122}$, i.e. **more than 120 significant digits agree**.

A PSLQ test of $\chi_2(1/\sqrt{2})$ against $\{\pi^2, \ln^2 2, L^2, L\ln 2, G, \pi L, \pi \ln 2, 1\}$ returned only a spurious relation (coefficients $\sim 10^5$, residual $\sim 10^{-45}$ at 120 digits), consistent with $\chi_2(1/\sqrt{2})$ being an independent constant: the dilogarithmic term cannot be removed in favor of these elementary constants.

Notes

1. **Equivalent forms.** The Landen-type identity

$$\chi_2\!\left(\frac{1-x}{1+x}\right) + \chi_2(x) = \frac{\pi^2}{8} - \frac{1}{2}\ln x\,\ln\frac{1-x}{1+x} \qquad (0 < x < 1)$$

is proved by differentiating: using $\chi_2'(z) = \frac{\operatorname{artanh} z}{z}$ and $\operatorname{artanh}\frac{1-x}{1+x} = -\frac{1}{2}\ln x$, the derivative of the left side is $\frac{\ln x}{1-x^2} + \frac{1}{2x}\ln\frac{1+x}{1-x}$, which matches the derivative of the right side; both sides agree in the limit $x \to 1^-$ (each tends to $\chi_2(0) + \chi_2(1) = \frac{\pi^2}{8}$, since $\ln x\,\ln\frac{1-x}{1+x} \to 0$). Taking $x = \frac{1}{\sqrt{2}}$, where $\frac{1-x}{1+x} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$, gives

$$\chi_2\!\left(\tfrac{1}{\sqrt{2}}\right) = \frac{\pi^2}{8} - \frac{\ln 2\,\ln(1+\sqrt{2})}{2} - \chi_2\!\left(3 - 2\sqrt{2}\right),$$

which yields the alternative expression for I quoted under **Result** (the $\sqrt{2}\,\pi^2$ terms cancel there). One may of course also write $\chi_2(z) = \frac{1}{2}[\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)]$ everywhere.

2. **Rigor.** The only analytic interchanges used are: (i) Tonelli for the nonnegative double integral in Step 4a; (ii) differentiation under the integral sign in Step 6d, justified by the uniform bound $\sqrt{2}$ on the derivative kernel; (iii) termwise integration of a power series inside its disc of convergence. All substitutions are monotone C^1 bijections, all improper endpoints are absolutely integrable, and every arctan/arcsin branch choice is fixed explicitly ($\phi \in [0, \frac{\pi}{2}]$ in Step 6a; the antiderivative v in Step 5 is continuous on $[0, 1]$).

3. **Irreducibility caveat.** The constant $\chi_2(1/\sqrt{2})$ (equivalently $\mathrm{Li}_2(\frac{\sqrt{2}}{2}) - \mathrm{Li}_2(-\frac{\sqrt{2}}{2})$, or $\chi_2(3 - 2\sqrt{2})$) is not known to reduce to π^2, $\ln^2 2$, $\ln^2(1+\sqrt{2})$, $\ln 2\ln(1+\sqrt{2})$, Catalan's constant, etc.; the PSLQ experiment above supports treating it as a genuinely new constant in the answer. This is expected: $1/\sqrt{2}$ lies on no known dilogarithm ladder over $\mathbb{Q}$.