

Cleo Bench Problem 16

$$\text{Integral } \int_0^1 \frac{\log(x) \log(1+x)}{\sqrt{1-x}} dx$$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^1 \frac{\log(x) \log(1+x)}{\sqrt{1-x}} dx.$$

(The original poster had expressed I through parameter-derivatives of ${}_2F_1$ at $(1, 0; \frac{3}{2}; -1)$ and asked for an explicit simplification.)

Result

With $L := \ln(1 + \sqrt{2})$,

$$I = 16 - 8 \ln 2 + \frac{\sqrt{2} \pi^2}{3} - 8\sqrt{2} \ln(1 + \sqrt{2}) + 4 \ln^2(1 + \sqrt{2}) + 16\sqrt{2} \operatorname{Li}_2(1 - \sqrt{2})$$

Numerically $I = -0.30458939446025561835671568624433462034460237466724 \dots$

Equivalent forms (obtained from the identities proved below, both verified numerically):

$$I = 16 - 8 \ln 2 - 8\sqrt{2} L + 4L^2 + 8\sqrt{2} L^2 - \frac{5\sqrt{2} \pi^2}{3} + 16\sqrt{2} \operatorname{Li}_2(\sqrt{2} - 1),$$

$$I = 16 - 8 \ln 2 - 8\sqrt{2} L + 4L^2 + 4\sqrt{2} L^2 - \frac{2\sqrt{2} \pi^2}{3} + 4\sqrt{2} \operatorname{Li}_2(3 - 2\sqrt{2}).$$

The single remaining dilogarithm cannot (as far as is known, and as PSLQ confirms numerically) be reduced further: only the *odd* part $\operatorname{Li}_2(\sqrt{2} - 1) - \operatorname{Li}_2(1 - \sqrt{2})$ is classically elementary (Landen), and it is used below; the even part is genuinely new.

Derivation

Throughout, Li_2 denotes the dilogarithm, defined for real $z \leq 1$ by

$$\operatorname{Li}_2(z) = - \int_0^z \frac{\ln(1-s)}{s} ds,$$

*Problem originally posed on Mathematics Stack Exchange ([question 1372767](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

which is real-analytic on $(-\infty, 1)$, agrees with $\sum_{n \geq 1} z^n/n^2$ for $|z| \leq 1$, and satisfies $\text{Li}'_2(z) = -\ln(1-z)/z$. We freely use $\text{Li}_2(1) = \zeta(2) = \pi^2/6$. Write $L := \ln(1+\sqrt{2})$; note $\ln(\sqrt{2}-1) = -L$ and $\ln(2-\sqrt{2}) = \frac{1}{2}\ln 2 - L$ (because $2-\sqrt{2} = \sqrt{2}(\sqrt{2}-1)$).

Convergence. Near $x = 0$ the integrand is $\sim x \ln x$, integrable; near $x = 1$ it is $O((1-x)^{1/2})$ because $\log x \rightarrow 0$. So I converges absolutely.

Step 1: remove the square root

Substitute $x = 1 - t^2$, $t = \sqrt{1-x} \in (0, 1)$, $dx = -2t dt$, so $dx/\sqrt{1-x} = -2 dt$:

$$I = 2 \int_0^1 \ln(1-t^2) \ln(2-t^2) dt.$$

Step 2: factor the logarithms

For $0 < t < 1$ all four factors below are positive, so

$$\ln(1-t^2) = \ln(1-t) + \ln(1+t), \quad \ln(2-t^2) = \ln(\sqrt{2}-t) + \ln(\sqrt{2}+t),$$

and therefore $I = 2(A_{--} + A_{-+} + A_{+-} + A_{++})$ with

$$A_{\sigma\tau} = \int_0^1 \ln(1+\sigma t) \ln(\sqrt{2}+\tau t) dt, \quad \sigma, \tau \in \{-, +\}.$$

Each $A_{\sigma\tau}$ converges absolutely ($\ln(1-t)$ is integrable). Shifting the variable ($u = 1-t$ for $\sigma = -$, $u = 1+t$ for $\sigma = +$) puts them in a uniform shape:

$$\begin{aligned} A_{--} &= \int_0^1 \ln u \ln(c_- + u) du, & A_{-+} &= \int_0^1 \ln u \ln(c_+ - u) du, \\ A_{++} &= \int_1^2 \ln u \ln(c_- + u) du, & A_{+-} &= \int_1^2 \ln u \ln(c_+ - u) du, \end{aligned}$$

where

$$c_- := \sqrt{2} - 1, \quad c_+ := \sqrt{2} + 1.$$

(For A_{-+} : $\sqrt{2}+t = \sqrt{2}+1-u = c_+ - u$; for A_{+-} : $\sqrt{2}-t = \sqrt{2}+1-u = c_+ - u$; for A_{++} : $\sqrt{2}+t = c_- + u$. All log arguments stay positive: $c_+ - u \geq c_+ - 2 = \sqrt{2} - 1 > 0$.)

Step 3: dilogarithm primitives

Lemma 1. For $c > 0$ and $0 \leq p < q$:

$$\int_p^q \frac{\ln(1+u/c)}{u} du = \text{Li}_2(-p/c) - \text{Li}_2(-q/c). \quad (\text{L1a})$$

For $c > q$ (so the argument stays in $(0, 1)$):

$$\int_p^q \frac{\ln(1-u/c)}{u} du = \text{Li}_2(p/c) - \text{Li}_2(q/c). \quad (\text{L1b})$$

Proof. Substitute $s = \mp u/c$ in the defining integral of Li_2 ; equivalently, $\frac{d}{du} \text{Li}_2(\mp u/c) = -\frac{\ln(1\pm u/c)}{u}$ by the chain rule. The integrands extend continuously to $u = 0$ (limit $\pm 1/c$), so $p = 0$ is allowed. ■

Note that (L1a) with $q/c > 1$ produces Li_2 at arguments < -1 ; this is exactly the real-analytic continuation defined above, so no branch issues arise.

Step 4: the four integrals

We use integration by parts, always differentiating $\ln u$. Let

$$V_c(u) := (c+u)\ln(c+u) - u, \quad W_c(u) := -(c-u)\ln(c-u) - u,$$

so $V'_c(u) = \ln(c+u)$ and $W'_c(u) = \ln(c-u)$. Two algebraic splittings will be used:

$$\frac{V_c(u) - V_c(0)}{u} = \frac{c\ln(1+u/c)}{u} + \ln(c+u) - 1, \quad \frac{W_c(u) - W_c(0)}{u} = -\frac{c\ln(1-u/c)}{u} + \ln(c-u) - 1. \quad (4.1)$$

(Check: $(c+u)\ln(c+u) - c\ln c = c\ln(1+u/c) + u\ln(c+u)$, and similarly for W .)

(F1) $\int_0^1 \ln u \ln(c+u) du$ ($c > 0$). Integrate by parts on $[\varepsilon, 1]$ with $v(u) = V_c(u) - V_c(0)$ (so $v(0) = 0$, $v' = \ln(c+u)$). Since $v(u) = \int_0^u \ln(c+w) dw = O(u)$, the boundary term $\ln \varepsilon v(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$, and $\ln 1 = 0$ kills the other end; also $v(u)/u$ is bounded, so the limit passes into the (absolutely convergent) integrals. Using (4.1), (L1a) with $(p, q) = (0, 1)$, and $\int_0^1 \ln(c+u) du = V_c(1) - V_c(0)$:

$$\int_0^1 \ln u \ln(c+u) du = -\int_0^1 \frac{v(u)}{u} du = 2 + c \ln c - (1+c) \ln(1+c) + c \operatorname{Li}_2(-1/c).$$

(F2) $\int_0^1 \ln u \ln(c-u) du$ ($c > 1$). Identically, with $v(u) = W_c(u) - W_c(0)$ and (L1b):

$$\int_0^1 \ln u \ln(c-u) du = 2 - c \ln c + (c-1) \ln(c-1) - c \operatorname{Li}_2(1/c).$$

(F3) $\int_1^2 \ln u \ln(c+u) du$ ($c > 0$). No singularities here; integrate by parts with V_c directly, then use (4.1) (the extra term $c \ln c/u$ integrates to $c \ln c \ln 2$) and (L1a) with $(p, q) = (1, 2)$:

$$\int_1^2 \ln u \ln(c+u) du = \ln 2 V_c(2) - V_c(2) + V_c(1) + 1 - c \ln c \ln 2 - c[\operatorname{Li}_2(-1/c) - \operatorname{Li}_2(-2/c)].$$

(F4) $\int_1^2 \ln u \ln(c-u) du$ ($c > 2$). Same with W_c and (L1b):

$$\int_1^2 \ln u \ln(c-u) du = \ln 2 W_c(2) - W_c(2) + W_c(1) + 1 + c \ln c \ln 2 + c[\operatorname{Li}_2(1/c) - \operatorname{Li}_2(2/c)].$$

Now insert $c = c_{\mp}$. Abbreviate the four dilogarithm values

$$D_1 = \operatorname{Li}_2(\sqrt{2}-1), \quad D_2 = \operatorname{Li}_2(2(\sqrt{2}-1)), \quad D_3 = \operatorname{Li}_2(-(\sqrt{2}+1)), \quad D_4 = \operatorname{Li}_2(-2(\sqrt{2}+1)),$$

noting $1/c_+ = c_- = \sqrt{2}-1$ and $1/c_- = c_+ = \sqrt{2}+1$. Using $c_- \ln c_- = -(\sqrt{2}-1)L$, $c_+ \ln c_+ = (\sqrt{2}+1)L$, $(1+c_-) \ln(1+c_-) = \frac{\sqrt{2}}{2} \ln 2 = (c_+ - 1) \ln(c_+ - 1)$, $V_{c_-}(2) = (\sqrt{2}+1)L - 2$, $V_{c_-}(1) = \frac{\sqrt{2}}{2} \ln 2 - 1$, $W_{c_+}(2) = (\sqrt{2}-1)L - 2$, $W_{c_+}(1) = -\frac{\sqrt{2}}{2} \ln 2 - 1$:

$$A_{--} = 2 - (\sqrt{2}-1)L - \frac{\sqrt{2}}{2} \ln 2 + (\sqrt{2}-1) D_3,$$

$$A_{-+} = 2 - (\sqrt{2}+1)L + \frac{\sqrt{2}}{2} \ln 2 - (\sqrt{2}+1) D_1,$$

$$A_{++} = 2 + 2\sqrt{2}L \ln 2 + \left(\frac{\sqrt{2}}{2} - 2\right) \ln 2 - (\sqrt{2}+1)L - (\sqrt{2}-1) D_3 + (\sqrt{2}-1) D_4,$$

$$A_{+-} = 2 + 2\sqrt{2}L \ln 2 - \left(2 + \frac{\sqrt{2}}{2}\right) \ln 2 - (\sqrt{2}-1)L + (\sqrt{2}+1) D_1 - (\sqrt{2}+1) D_2.$$

(Each of these four closed forms was verified numerically to 80 digits against direct quadrature of the corresponding integral.)

Step 5: assemble

Adding, the D_1 and D_3 terms cancel **exactly**, the $\frac{\sqrt{2}}{2} \ln 2$ terms cancel, and

$$A_{--} + A_{-+} + A_{+-} + A_{++} = 8 - 4 \ln 2 - 4\sqrt{2}L + 4\sqrt{2}L \ln 2 - (\sqrt{2} + 1)D_2 + (\sqrt{2} - 1)D_4,$$

hence

$$I = 16 - 8 \ln 2 - 8\sqrt{2}L + 8\sqrt{2}L \ln 2 - 2(\sqrt{2} + 1)D_2 + 2(\sqrt{2} - 1)D_4. \quad (5.1)$$

Step 6: dilogarithm identities

Lemma 2 (Landen). For $0 < x < 1$:

$$\operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{-x}{1-x}\right) = -\frac{1}{2} \ln^2(1-x).$$

Proof. Both sides vanish as $x \rightarrow 0^+$ and are C^1 on $(0, 1)$ (the second argument runs through $(-\infty, 0)$, where Li_2 is the real-analytic function above). Differentiating, with $z = \frac{-x}{1-x}$, $1-z = \frac{1}{1-x}$, $z' = -\frac{1}{(1-x)^2}$:

$$-\frac{\ln(1-x)}{x} + \left(-\frac{\ln(1-z)}{z}\right)z' = -\frac{\ln(1-x)}{x} + \frac{\ln(1-x)}{x(1-x)} = \frac{\ln(1-x)}{1-x},$$

which equals $\frac{d}{dx}[-\frac{1}{2} \ln^2(1-x)]$. ■

Lemma 3 (Euler reflection). For $0 < x < 1$: $\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x)$.

Proof. Differentiate: both sides have derivative $-\frac{\ln(1-x)}{x} + \frac{\ln x}{1-x}$; as $x \rightarrow 0^+$ the left side $\rightarrow \operatorname{Li}_2(1) = \pi^2/6$ and $\ln x \ln(1-x) \rightarrow 0$. ■

Lemma 4 (squaring). For $|x| \leq 1$: $\operatorname{Li}_2(x) + \operatorname{Li}_2(-x) = \frac{1}{2} \operatorname{Li}_2(x^2)$ (immediate from the absolutely convergent series). In particular, with $x = \frac{1}{2}$ in Lemma 3, $\operatorname{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 2$.

Lemma 5 (Landen's value at $\sqrt{2}-1$).

$$\operatorname{Li}_2(\sqrt{2}-1) - \operatorname{Li}_2(1-\sqrt{2}) = \frac{\pi^2}{8} - \frac{L^2}{2}.$$

Proof. Apply Lemma 2 at $x_1 = 1 - \frac{1}{\sqrt{2}}$ (then $\frac{-x_1}{1-x_1} = 1 - \sqrt{2}$) and at $x_2 = \sqrt{2}-1$ (then $\frac{-x_2}{1-x_2} = -\frac{1}{\sqrt{2}}$, since $1-x_2 = 2 - \sqrt{2} = \sqrt{2}(\sqrt{2}-1)$):

$$\operatorname{Li}_2(1 - \frac{1}{\sqrt{2}}) + \operatorname{Li}_2(1 - \sqrt{2}) = -\frac{1}{8} \ln^2 2, \quad \operatorname{Li}_2(\sqrt{2}-1) + \operatorname{Li}_2(-\frac{1}{\sqrt{2}}) = -\frac{1}{2}(\frac{1}{2} \ln 2 - L)^2.$$

Lemma 3 at $x = \frac{1}{\sqrt{2}}$ (with $\ln(1 - \frac{1}{\sqrt{2}}) = -L - \frac{1}{2} \ln 2$) and Lemma 4 at $x = \frac{1}{\sqrt{2}}$ give

$$\operatorname{Li}_2(\frac{1}{\sqrt{2}}) + \operatorname{Li}_2(1 - \frac{1}{\sqrt{2}}) = \frac{\pi^2}{6} - \frac{L \ln 2}{2} - \frac{\ln^2 2}{4}, \quad \operatorname{Li}_2(\frac{1}{\sqrt{2}}) + \operatorname{Li}_2(-\frac{1}{\sqrt{2}}) = \frac{\pi^2}{24} - \frac{\ln^2 2}{4}.$$

Forming (second) - (fourth) + (third) - (first), all dilogarithms except the two desired ones cancel, and the right side collapses to $\frac{\pi^2}{8} - \frac{L^2}{2}$. ■

Reduction of D_2, D_4 . Apply Lemma 2 at $x = 2(\sqrt{2}-1) \in (0, 1)$: here

$$1-x = 3 - 2\sqrt{2} = (\sqrt{2}-1)^2, \quad \frac{-x}{1-x} = \frac{-2(\sqrt{2}-1)}{(\sqrt{2}-1)^2} = -\frac{2}{\sqrt{2}-1} = -2(\sqrt{2}+1),$$

so

$$D_2 + D_4 = -\frac{1}{2} \ln^2((\sqrt{2}-1)^2) = -2L^2. \quad (6.1)$$

Next, Lemma 3 at the same x (with $\ln x = \ln 2 - L$, $\ln(1-x) = -2L$), Lemma 4 at $x = \sqrt{2} - 1$, and Lemma 5 give

$$D_2 = \frac{\pi^2}{6} + 2L \ln 2 - 2L^2 - \operatorname{Li}_2((\sqrt{2} - 1)^2), \quad \operatorname{Li}_2((\sqrt{2} - 1)^2) = 4 \operatorname{Li}_2(1 - \sqrt{2}) + \frac{\pi^2}{4} - L^2,$$

hence

$$D_2 = -\frac{\pi^2}{12} + 2L \ln 2 - L^2 - 4 \operatorname{Li}_2(1 - \sqrt{2}). \quad (6.2)$$

Step 7: final reduction

Insert $D_4 = -2L^2 - D_2$ from (6.1) into (5.1):

$$I = 16 - 8 \ln 2 - 8\sqrt{2}L + 8\sqrt{2}L \ln 2 - 4(\sqrt{2} - 1)L^2 - 4\sqrt{2}D_2,$$

then substitute (6.2). The $L \ln 2$ terms cancel ($8\sqrt{2} - 8\sqrt{2}$) and the L^2 terms combine ($-4\sqrt{2} + 4 + 4\sqrt{2} = 4$):

$$I = 16 - 8 \ln 2 + \frac{\sqrt{2}\pi^2}{3} - 8\sqrt{2} \ln(1 + \sqrt{2}) + 4 \ln^2(1 + \sqrt{2}) + 16\sqrt{2} \operatorname{Li}_2(1 - \sqrt{2}).$$

The two equivalent forms in the Result section follow by rewriting $\operatorname{Li}_2(1 - \sqrt{2})$ via Lemma 5 ($\operatorname{Li}_2(1 - \sqrt{2}) = \operatorname{Li}_2(\sqrt{2} - 1) - \frac{\pi^2}{8} + \frac{L^2}{2}$) or via Lemma 4/Lemma 5 ($4 \operatorname{Li}_2(1 - \sqrt{2}) = \operatorname{Li}_2(3 - 2\sqrt{2}) - \frac{\pi^2}{4} + L^2$).

Numerical verification

All computations with `mpmath` (scripts `verify1.py`, `verify2.py` in the scratch directory).

- Direct quadrature of the original integral (`tanh-sinh`, split at 0.5 and 0.99, `mp.dps=80`) and of the transformed integral $2 \int_0^1 \ln(1 - t^2) \ln(2 - t^2) dt$ agree to all 80 digits:

$$I = -0.304589394460255618356715686244334620344602374667238662810239 \dots$$

- Each of the four closed forms $A_{--}, A_{-+}, A_{+-}, A_{++}$ was checked against direct quadrature: errors $\leq 1.3 \cdot 10^{-80}$.
- Identities (6.1), (6.2) and Lemma 5 were each checked to $\sim 10^{-81}$.
- Final check at `mp.dps=100`: quadrature vs. the boxed closed form agree to **90 significant digits** (difference $1.8 \cdot 10^{-100}$, i.e. at quadrature precision):

$$\begin{aligned} & -0.30458939446025561835671568624433462034460237466723866281023 \\ & \quad 8932648405863027220081308329913\dots \end{aligned}$$

- Independent confirmation by integer-relation detection: PSLQ applied to $\{I, 1, \ln 2, \sqrt{2}L, L^2, \sqrt{2}\pi^2, \sqrt{2} \operatorname{Li}_2(1 - \sqrt{2}), \dots\}$ (60 digits) recovered exactly $-3I + 48 - 24 \ln 2 - 24\sqrt{2}L + 12L^2 + \sqrt{2}\pi^2 + 48\sqrt{2} \operatorname{Li}_2(1 - \sqrt{2}) = 0$, i.e. the boxed formula.
- Cross-check of the problem statement: the poster's expression $4(1 - \ln 2) {}_2F_1^{(0,1,0,0)}(1, 0; \frac{3}{2}; -1) - 2 {}_2F_1^{(1,1,0,0)}(1, 0; \frac{3}{2}; -1) - 2 {}_2F_1^{(0,1,1,0)}(1, 0; \frac{3}{2}; -1)$, evaluated with `mpmath` parameter differentiation, matches I to ~ 41 digits.

Notes

- The derivation is complete and self-contained: substitution, four elementary-by-parts integrals with the dilogarithm primitive (Lemma 1), and the classical dilogarithm identities (Lemmas 2–5), each proved by differentiation from $\text{Li}_2(z) = -\int_0^z \ln(1-s) ds/s$ plus $\zeta(2) = \pi^2/6$ (Basel; assumed known). All Li_2 arguments stay on $(-\infty, 1)$, where Li_2 is real-analytic, so there are no branch subtleties.
- One dilogarithm, $\text{Li}_2(1 - \sqrt{2})$ (equivalently $\text{Li}_2(\sqrt{2} - 1)$ or $\text{Li}_2(3 - 2\sqrt{2})$), remains in the answer. Only the odd combination $\text{Li}_2(\sqrt{2} - 1) - \text{Li}_2(1 - \sqrt{2}) = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(1 + \sqrt{2})$ is elementary (Lemma 5); the even combination is not expected to reduce. As heuristic support, PSLQ at 60 digits with coefficient bound 10^8 found **no** rational relation between $\text{Li}_2(1 - \sqrt{2})$ and $\{\pi^2, \ln^2 2, L^2, L \ln 2, 1\}$. So this is the natural “simplest” closed form; a proof of irreducibility is of course not claimed (such independence statements are open in general).
- Caveat none otherwise: every algebraic step above was additionally verified numerically to ≥ 80 digits.