

## Cleo Bench Problem 8

Integrals of the form  $\int_0^\infty \operatorname{arccot}(x) \cdot \operatorname{arccot}(ax) \cdot \operatorname{arccot}(bx) dx$

Derivation by Claude (Fable 5), closed-book\*

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**Solution:**  $I(a, b) = \int_0^\infty \operatorname{arccot}(x) \operatorname{arccot}(ax) \operatorname{arccot}(bx) dx$  — **the case**  
 $b = 1$

### Problem

For  $a, b > 0$  let

$$I(a, b) = \int_0^\infty \operatorname{arccot}(x) \cdot \operatorname{arccot}(ax) \cdot \operatorname{arccot}(bx) dx.$$

Known are  $I(a, 0) = \frac{\pi^2}{4} \left[ \ln\left(1 + \frac{1}{a}\right) + \frac{\ln(1+a)}{a} \right]$  (G&R 4.511) and  $I(1, 1) = \frac{3\pi^2}{4} \ln 2 - \frac{21}{8} \zeta(3)$ .

**Asked:** a general closed form for  $I(a, 1)$ , or at least closed forms for  $I(2, 1)$  and  $I(3, 1)$ .

Throughout,  $\operatorname{arccot} x := \arctan(1/x) \in (0, \pi/2)$  for  $x > 0$ , and

$$\chi_s(z) := \frac{1}{2} [\operatorname{Li}_s(z) - \operatorname{Li}_s(-z)] = \sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)^s} \quad (\text{Legendre chi}).$$

### Result

**General closed form.** For every  $a \geq 1$ , with

$$c = \frac{a-1}{a+1}, \quad 1+c = \frac{2a}{a+1}, \quad 1-c^2 = \frac{4a}{(a+1)^2},$$

$$\begin{aligned} I(a, 1) = & \frac{\pi^2}{4a} \left[ (a+1) \ln(a+1) - a \ln a + a \ln 2 \right] + \frac{7(a-1)}{8a} \zeta(3) \\ & + \frac{a-1}{a} \chi_3\left(\frac{a-1}{a+1}\right) - \chi_3\left(\frac{1}{a}\right) - \ln a \chi_2\left(\frac{1}{a}\right) - 2 \ln \frac{2a}{a+1} \chi_2\left(\frac{a-1}{a+1}\right) \\ & + \ln a \ln \frac{2a}{a+1} \ln \frac{a-1}{a+1} - \frac{2}{3} \ln^3 \frac{2a}{a+1} \\ & + \frac{1}{4} \operatorname{Li}_3\left(\frac{4a}{(a+1)^2}\right) - \frac{1}{4} \ln \frac{4a}{(a+1)^2} \operatorname{Li}_2\left(\frac{4a}{(a+1)^2}\right) \\ & - 2 \operatorname{Li}_3\left(\frac{a+1}{2a}\right) - 2 \ln \frac{2a}{a+1} \operatorname{Li}_2\left(\frac{a+1}{2a}\right). \end{aligned}$$

(At  $a = 1$  the term with  $\ln \frac{a-1}{a+1}$  is absent — its coefficient vanishes — and the formula evaluates *exactly* to  $\frac{3\pi^2}{4} \ln 2 - \frac{21}{8} \zeta(3)$ , reproducing (3); see §7.)

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\*Problem originally posed on Mathematics Stack Exchange ([question 1279165](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

**Companion for  $0 < a < 1$**  (same method, stated for completeness; §8):

$$\begin{aligned}
I(a, 1) &= \frac{\pi^2}{4a} \left[ (a+1) \ln(a+1) - a \ln a + a \ln 2 \right] + \frac{7(a-1)}{8a} \zeta(3) \\
&\quad + \frac{1-a}{a} \chi_3\left(\frac{1-a}{1+a}\right) - \chi_3(a) + \ln a \chi_2(a) + 2 \ln \frac{2a}{1+a} \chi_2\left(\frac{1-a}{1+a}\right) \\
&\quad + \ln a \ln \frac{2a}{1+a} \ln \frac{1-a}{1+a} + \frac{1}{4} \operatorname{Li}_3\left(\frac{4a}{(1+a)^2}\right) - \frac{1}{4} \ln \frac{4a}{(1+a)^2} \operatorname{Li}_2\left(\frac{4a}{(1+a)^2}\right) \\
&\quad - 2 \operatorname{Li}_3\left(\frac{2a}{1+a}\right) + 2 \ln \frac{2a}{1+a} \operatorname{Li}_2\left(\frac{2a}{1+a}\right).
\end{aligned}$$

**The two requested special values.** Reducing all polylogarithms by classical identities (§9):

$$\begin{aligned}
I(2, 1) &= \frac{21}{4} \zeta(3) + \frac{17}{2} \operatorname{Li}_3\left(-\frac{1}{2}\right) - \frac{3}{4} \operatorname{Li}_3\left(\frac{1}{3}\right) + \frac{3}{4} \operatorname{Li}_3\left(-\frac{1}{3}\right) - \frac{1}{4} \operatorname{Li}_3\left(-\frac{1}{8}\right) \\
&\quad - 2 \ln 2 \operatorname{Li}_2\left(\frac{1}{3}\right) + \pi^2 \left( \frac{3 \ln 3}{8} - \frac{7 \ln 2}{12} \right) + 2 \ln^2 2 \ln 3 - \ln 2 \ln^2 3 - \frac{7}{24} \ln^3 2
\end{aligned}$$

$$\begin{aligned}
I(3, 1) &= -\frac{7}{4} \zeta(3) + \frac{3}{2} \operatorname{Li}_3\left(\frac{1}{3}\right) + \frac{1}{4} \operatorname{Li}_3\left(-\frac{1}{3}\right) + \frac{2}{3} \operatorname{Li}_3\left(-\frac{1}{2}\right) \\
&\quad + \ln 3 \operatorname{Li}_2\left(\frac{1}{3}\right) + \pi^2 \left( \frac{35 \ln 2}{36} - \frac{3 \ln 3}{8} \right) - \frac{\ln^3 2}{9} + \frac{5 \ln^3 3}{24}
\end{aligned}$$

Numerically,

$$I(2, 1) = 1.51208605444427919260956727012207121505106065118750693166259359404 \dots$$

$$I(3, 1) = 1.25096386740952300916042368441067490079970979116162141329074248875 \dots$$

A by-product worth displaying is the central lemma (Step 3), of independent interest:

$$\int_0^\infty \frac{x \operatorname{arccot}^2 x}{x^2 + \gamma^2} dx = \frac{\pi^2}{4} \ln \frac{1+\gamma}{\gamma} - \frac{7}{8} \zeta(3) + \chi_3\left(\frac{\gamma-1}{\gamma+1}\right) \quad (\gamma > 0).$$

## Derivation

### 0. Plan and notation

Since two factors coincide when  $b = 1$ ,

$$I(a, 1) = \int_0^\infty \operatorname{arccot}^2(x) \operatorname{arccot}(ax) dx.$$

We (1) differentiate in the parameter to reduce to the one-parameter integral  $N(\gamma)$  above; (2) evaluate  $N(\gamma)$  in closed form (Key Lemma); (3) integrate  $N$  once more, which requires four classical weight-3 integrals with explicit antiderivatives; (4) assemble, and (5) specialize to  $a = 2, 3$ . Every antiderivative below can be checked by direct differentiation (all were also machine-verified symbolically, and every displayed identity was verified numerically to  $\geq 60$  digits).

We use repeatedly:  $\frac{d}{dz} \text{Li}_2(z) = -\frac{\ln(1-z)}{z}$ ,  $\frac{d}{dz} \text{Li}_3(z) = \frac{\text{Li}_2(z)}{z}$ , hence

$$\begin{aligned} \frac{d}{dz} \text{Li}_2(1-z) &= \frac{\ln z}{1-z}, & \frac{d}{dz} \text{Li}_3(1-z) &= -\frac{\text{Li}_2(1-z)}{1-z}, \\ \chi'_2(z) &= \frac{\text{artanh } z}{z}, & \chi'_3(z) &= \frac{\chi_2(z)}{z}. \end{aligned}$$

Also  $\text{Li}_2(1) = \frac{\pi^2}{6}$ ,  $\text{Li}_2(-1) = -\frac{\pi^2}{12}$ ,  $\chi_2(1) = \frac{\pi^2}{8}$ ,  $\chi_3(1) = \sum_{k \geq 0} (2k+1)^{-3} = (1-2^{-3})\zeta(3) = \frac{7}{8}\zeta(3)$ .

## 1. Reduction to one parameter

For  $\gamma \geq 0$  put

$$\Phi(\gamma) := \int_0^\infty \text{arccot}^2(x) \arctan \frac{\gamma}{x} dx, \quad \text{so} \quad I(a, 1) = \Phi(1/a), \quad \Phi(0) = 0,$$

because  $\text{arccot}(ax) = \arctan \frac{1/a}{x}$ .

For  $0 < \gamma_0 \leq \gamma \leq \Gamma$  the  $\gamma$ -derivative of the integrand is  $\text{arccot}^2(x) \frac{x}{x^2+\gamma^2}$ , dominated by the integrable function  $\text{arccot}^2(x) \frac{x}{x^2+\gamma_0^2} = O(\min(x, x^{-3}))$ ; hence  $\Phi$  is differentiable on  $(0, \infty)$  with

$$\Phi'(\gamma) = N(\gamma) := \int_0^\infty \frac{x \text{arccot}^2 x}{x^2 + \gamma^2} dx \geq 0.$$

By dominated convergence  $\Phi$  is continuous at  $0^+$ , so by monotone convergence and the fundamental theorem of calculus,

$$I(a, 1) = \int_0^{1/a} N(\gamma) d\gamma. \quad (1.1)$$

## 2. The auxiliary function $\Lambda$

Let

$$\Lambda(z) := \int_0^z \frac{-\ln s}{1-s^2} ds \quad (z > 0),$$

whose integrand is positive and continuous on  $(0, \infty)$  (removable singularity at  $s = 1$ , where it tends to  $\frac{1}{2}$ ).

$$(2a) \quad \Lambda(1) = \int_0^1 (-\ln s) \sum_{k \geq 0} s^{2k} ds = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \text{ (monotone convergence).}$$

(2b) *Landen form*: for all  $z > 0$ ,

$$\Lambda(z) = \frac{\pi^2}{8} + \chi_2\left(\frac{z-1}{z+1}\right). \quad (2.1)$$

*Proof.* Both sides equal  $\frac{\pi^2}{8}$  at  $z = 1$ . With  $u = \frac{z-1}{z+1} \in (-1, 1)$  one has  $\text{artanh } u = \frac{1}{2} \ln \frac{1+u}{1-u} = \frac{1}{2} \ln z$  and  $u'(z) = \frac{2}{(z+1)^2}$ , so

$$\frac{d}{dz} \chi_2(u) = \frac{\text{artanh } u}{u} u' = \frac{\ln z}{2} \cdot \frac{z+1}{z-1} \cdot \frac{2}{(z+1)^2} = \frac{\ln z}{z^2-1} = -\frac{\ln z}{1-z^2} = \Lambda'(z). \quad \square$$

(2c) For  $p, \eta > 0$ :

$$A(p, \eta) := \int_0^\infty \frac{\arctan(\eta/x)}{x^2 + p^2} dx = \frac{1}{p} \Lambda\left(\frac{\eta}{p}\right). \quad (2.2)$$

*Proof.* Scaling  $x = p\xi$  reduces to  $p = 1$ . Let  $\Lambda_0(\mu) = \int_0^\infty \frac{\arctan(\mu/\xi)}{1+\xi^2} d\xi$ ; then  $\Lambda_0(0) = 0$  and (differentiation under the integral, justified as in §1)

$$\begin{aligned}\Lambda_0'(\mu) &= \int_0^\infty \frac{\xi d\xi}{(\xi^2 + \mu^2)(1 + \xi^2)} = \frac{1}{2} \int_0^\infty \frac{du}{(u + \mu^2)(u + 1)} \\ &= \frac{1}{2(1 - \mu^2)} \left[ \ln \frac{u + \mu^2}{u + 1} \right]_0^\infty = \frac{-\ln \mu}{1 - \mu^2} = \Lambda'(\mu),\end{aligned}$$

so  $\Lambda_0 = \Lambda$ .  $\square$

### 3. Key Lemma

**Lemma 1.** *For every  $\gamma > 0$ ,*

$$N(\gamma) = \int_0^\infty \frac{x \operatorname{arccot}^2 x}{x^2 + \gamma^2} dx = \frac{\pi^2}{4} \ln \frac{1 + \gamma}{\gamma} - \frac{7}{8} \zeta(3) + \chi_3 \left( \frac{\gamma - 1}{\gamma + 1} \right).$$

*Proof.* By the fundamental theorem of calculus in  $s$ ,

$$\operatorname{arccot}^2 x = \arctan^2 \frac{1}{x} = 2 \int_0^1 \arctan \frac{s}{x} \cdot \frac{x}{x^2 + s^2} ds.$$

All integrands are nonnegative, so Tonelli gives

$$N(\gamma) = 2 \int_0^1 \int_0^\infty \frac{x^2 \arctan(s/x)}{(x^2 + s^2)(x^2 + \gamma^2)} dx ds.$$

For  $s \neq \gamma$ ,  $\frac{x^2}{(x^2 + s^2)(x^2 + \gamma^2)} = \frac{1}{\gamma^2 - s^2} \left[ \frac{\gamma^2}{x^2 + \gamma^2} - \frac{s^2}{x^2 + s^2} \right]$ , so by (2.2) and  $\Lambda(1) = \frac{\pi^2}{8}$  the inner integral equals

$$\frac{\gamma^2 A(\gamma, s) - s^2 A(s, s)}{\gamma^2 - s^2} = \frac{\gamma \Lambda(s/\gamma) - \frac{\pi^2}{8} s}{\gamma^2 - s^2}.$$

The apparent pole at  $s = \gamma$  is removable (numerator vanishes there). Substituting  $s = \gamma\sigma$ ,

$$N(\gamma) = 2 \int_0^{1/\gamma} \frac{\Lambda(\sigma) - \frac{\pi^2}{8} \sigma}{1 - \sigma^2} d\sigma. \quad (3.1)$$

Insert (2.1) in the form  $\Lambda(\sigma) - \frac{\pi^2}{8} \sigma = \frac{\pi^2}{8} (1 - \sigma) - \chi_2 \left( \frac{1 - \sigma}{1 + \sigma} \right)$ :

$$N(\gamma) = \frac{\pi^2}{4} \int_0^{1/\gamma} \frac{d\sigma}{1 + \sigma} - 2 \int_0^{1/\gamma} \frac{\chi_2 \left( \frac{1 - \sigma}{1 + \sigma} \right)}{1 - \sigma^2} d\sigma.$$

In the second integral substitute  $u = \frac{1 - \sigma}{1 + \sigma}$ , i.e.  $\sigma = \frac{1 - u}{1 + u}$ , for which  $\frac{d\sigma}{1 - \sigma^2} = -\frac{du}{2u}$ ; as  $\sigma : 0 \rightarrow 1/\gamma$ ,  $u : 1 \rightarrow \frac{\gamma - 1}{\gamma + 1}$ . Using  $\chi_3'(u) = \chi_2(u)/u$ ,

$$2 \int_0^{1/\gamma} \frac{\chi_2(u(\sigma))}{1 - \sigma^2} d\sigma = \int_{(\gamma - 1)/(\gamma + 1)}^1 \frac{\chi_2(u)}{u} du = \chi_3(1) - \chi_3 \left( \frac{\gamma - 1}{\gamma + 1} \right).$$

With  $\chi_3(1) = \frac{7}{8} \zeta(3)$  the Lemma follows.  $\square$

(Check:  $\gamma = 1$  gives the classical  $\int_0^\infty \frac{x \operatorname{arccot}^2 x}{1 + x^2} dx = \int_0^{\pi/2} \theta^2 \cot \theta d\theta = \frac{\pi^2}{4} \ln 2 - \frac{7}{8} \zeta(3)$ .)

#### 4. Integrating the Lemma; reduction to $X(c)$

Let  $w = \frac{1}{a} \in (0, 1]$  (so  $a \geq 1$ ). From (1.1),

$$I(a, 1) = \frac{\pi^2}{4} \left[ (1+w) \ln(1+w) - w \ln w \right] - \frac{7}{8} \zeta(3) w + \Xi(w), \quad \Xi(w) := \int_0^w \chi_3 \left( \frac{\gamma-1}{\gamma+1} \right) d\gamma, \quad (4.1)$$

using  $\int_0^w \ln \frac{1+\gamma}{\gamma} d\gamma = (1+w) \ln(1+w) - w \ln w$ .

In  $\Xi$  substitute  $z = \frac{1-\gamma}{1+\gamma}$  ( $\gamma = \frac{1-z}{1+z}$ ,  $d\gamma = -\frac{2dz}{(1+z)^2}$ ) and use that  $\chi_3$  is odd:

$$\Xi(w) = -2 \int_c^1 \frac{\chi_3(z)}{(1+z)^2} dz, \quad c := \frac{1-w}{1+w} = \frac{a-1}{a+1} \in [0, 1).$$

Integrating by parts with  $d(-\frac{1}{1+z})$  and  $\chi_3'(z) = \chi_2(z)/z$ , then splitting  $\frac{1}{z(1+z)} = \frac{1}{z} - \frac{1}{1+z}$  and using  $\int_c^1 \frac{\chi_2}{z} = \chi_3(1) - \chi_3(c)$ :

$$\Xi(w) = -\frac{7}{8} \zeta(3) + \frac{2c}{1+c} \chi_3(c) + 2X(c), \quad X(c) := \int_c^1 \frac{\chi_2(z)}{1+z} dz, \quad \frac{2c}{1+c} = \frac{a-1}{a}. \quad (4.2)$$

#### 5. Evaluation of $X(c)$

Write  $\chi_2 = \frac{1}{2}(\text{Li}_2(z) - \text{Li}_2(-z))$  and integrate each part by parts against  $d \ln(1+z)$ :

$$\begin{aligned} \int_c^1 \frac{\text{Li}_2(z)}{1+z} dz &= \left[ \ln(1+z) \text{Li}_2(z) \right]_c^1 + \int_c^1 \frac{\ln(1+z) \ln(1-z)}{z} dz, \\ \int_c^1 \frac{\text{Li}_2(-z)}{1+z} dz &= \left[ \ln(1+z) \text{Li}_2(-z) \right]_c^1 + \int_c^1 \frac{\ln^2(1+z)}{z} dz. \end{aligned}$$

Since  $\text{Li}_2(1) - \text{Li}_2(-1) = \frac{\pi^2}{4}$ ,

$$X(c) = \frac{\pi^2 \ln 2}{8} - \ln(1+c) \chi_2(c) + \frac{1}{2} \mathcal{A}(c) - \frac{1}{2} \mathcal{B}(c), \quad (5.1)$$

$$\mathcal{A}(c) := \int_c^1 \frac{\ln(1-z) \ln(1+z)}{z} dz, \quad \mathcal{B}(c) := \int_c^1 \frac{\ln^2(1+z)}{z} dz.$$

#### 6. The four antiderivative lemmas

$$\text{(A1)} \quad \mathcal{C}(x) := \int_0^x \frac{\ln^2(1-v)}{v} dv = \ln^2(1-x) \ln x + 2 \ln(1-x) \text{Li}_2(1-x) - 2 \text{Li}_3(1-x) + 2\zeta(3).$$

*Proof:* differentiate the right side; all non-elementary terms cancel pairwise, leaving  $\frac{\ln^2(1-x)}{x}$ ; the right side  $\rightarrow 0$  as  $x \rightarrow 0^+$ . In particular  $\mathcal{C}(1) = 2\zeta(3)$ .

$\text{(A2)}$   $g(z) := \int_0^z \frac{\ln^2(1+t)}{t} dt = \ln^2(1+z) \ln z - \frac{2}{3} \ln^3(1+z) - 2 \ln(1+z) \text{Li}_2\left(\frac{1}{1+z}\right) - 2 \text{Li}_3\left(\frac{1}{1+z}\right) + 2\zeta(3)$ . *Proof:* substitute  $s = \frac{t}{1+t}$ , so  $\frac{dt}{t} = \frac{ds}{s(1-s)}$  and  $\ln(1+t) = -\ln(1-s)$ ; then  $g(z) = \mathcal{C}\left(\frac{z}{1+z}\right) + \frac{1}{3} \ln^3(1+z)$ , and expand (A1) with  $1 - \frac{z}{1+z} = \frac{1}{1+z}$ . (Equivalently: differentiate the right side directly.) With  $\text{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$  and  $\text{Li}_3(\frac{1}{2}) = \frac{7\zeta(3)}{8} - \frac{\pi^2 \ln 2}{12} + \frac{\ln^3 2}{6}$  one gets  $g(1) = \frac{\zeta(3)}{4}$ , hence

$$\mathcal{B}(c) = g(1) - g(c) = \frac{\zeta(3)}{4} - g(c). \quad (6.1)$$

$\text{(A3)}$  Let  $\Psi(t) := \ln^2 t \ln \frac{1-t}{1+t} - 4\chi_3(t) + 4 \ln t \chi_2(t)$ . Then, with  $t(z) = \frac{1-z}{1+z}$ ,

$$\frac{d}{dz} \Psi(t(z)) = \frac{1}{z} \ln^2 \frac{1-z}{1+z}, \quad \Psi(1) = -4\chi_3(1) = -\frac{7}{2} \zeta(3), \quad \Psi(0^+) = 0.$$

*Proof:* under  $t = \frac{1-z}{1+z}$  one has  $\frac{dz}{z} = -\frac{2dt}{1-t^2}$  and  $\ln \frac{1-z}{1+z} = \ln t$ , so the claim is equivalent to  $\int \frac{\ln^2 t}{1-t^2} dt$  having primitive  $-\frac{1}{2}\Psi(t)$ , which follows from  $\int \frac{\ln^2 t}{1-t^2} dt = \mp \ln^2 t \ln(1 \mp t) \pm 2[\text{Li}_3(\pm t) - \ln t \text{Li}_2(\pm t)]$ , each checked by differentiation using  $\frac{d}{dt}[\text{Li}_3(\pm t) - \ln t \text{Li}_2(\pm t)] = \frac{\ln t \ln(1 \mp t)}{t}$ .

**(A4)** From the algebraic identity  $\ln(1-z)\ln(1+z) = \frac{1}{4}\ln^2(1-z^2) - \frac{1}{4}\ln^2 \frac{1-z}{1+z}$  and  $v = z^2$  in the first part,

$$\int_0^c \frac{\ln(1-z)\ln(1+z)}{z} dz = \frac{1}{8}\mathcal{C}(c^2) - \frac{1}{4}[\Psi(t(c)) - \Psi(1)], \quad t(c) = \frac{1-c}{1+c} = \frac{1}{a}.$$

At  $c = 0$  this yields the classical  $\int_0^1 \frac{\ln(1-z)\ln(1+z)}{z} dz = \frac{\zeta(3)}{4} - \frac{7}{8}\zeta(3) = -\frac{5}{8}\zeta(3)$ , and consequently

$$\mathcal{A}(c) = \frac{\zeta(3)}{4} - \frac{1}{8}\mathcal{C}(c^2) + \frac{1}{4}\Psi\left(\frac{1}{a}\right). \quad (6.2)$$

## 7. Assembly: the general formula for $a \geq 1$

Insert (6.1)–(6.2) into (5.1), then (5.1) into (4.2), then (4.2) into (4.1). Collecting the  $\zeta(3)$ 's:  $-\frac{7w}{8} - \frac{7}{8} - \frac{1}{4} + 2 = \frac{7}{8}(1-w) = \frac{7(a-1)}{8a}$ . Note  $\Psi(1/a) = \ln^2 a \ln c - 4\chi_3(1/a) - 4\ln a \chi_2(1/a)$  (since  $\ln \frac{1-1/a}{1+1/a} = \ln c$ ), and  $1+c = \frac{2a}{a+1}$ ,  $1-c^2 = \frac{4a}{(a+1)^2}$ .

The pure-logarithm terms multiplying  $\ln c$  collapse: writing  $a = \frac{1+c}{1-c}$ , i.e.  $\ln a = \ln(1+c) - \ln(1-c)$ ,

$$\begin{aligned} & \frac{\ln^2 a - \ln^2(1-c^2)}{4} + \ln^2(1+c) \\ &= \frac{[\ln(1+c) - \ln(1-c)]^2 - [\ln(1+c) + \ln(1-c)]^2}{4} + \ln^2(1+c) = \ln a \ln(1+c). \end{aligned}$$

This yields exactly the boxed general formula in **Result**. Its evaluation at  $a = 1$  ( $c = 0$ ) is exact:

$$\begin{aligned} I(1, 1) &= \frac{\pi^2}{4}[2\ln 2 + \ln 2] - \chi_3(1) + \frac{1}{4}\text{Li}_3(1) - 2\text{Li}_3(1) \\ &= \frac{3\pi^2}{4}\ln 2 - \left(\frac{7}{8} - \frac{1}{4} + 2\right)\zeta(3) = \frac{3\pi^2}{4}\ln 2 - \frac{21}{8}\zeta(3), \end{aligned}$$

which is the known value (3) — a strong structural check.

## 8. The companion branch $0 < a < 1$

Here  $w = 1/a > 1$ . Split  $\Xi(w) = \Xi(1) + \int_1^w \chi_3(\frac{\gamma-1}{\gamma+1})d\gamma$  and substitute  $z = \frac{\gamma-1}{\gamma+1}$  (Jacobian now  $\frac{2}{(1-z)^2}$ ):

$$\Xi(w) = \Xi(1) + 2 \int_0^{c'} \frac{\chi_3(z)}{(1-z)^2} dz, \quad c' = \frac{1-a}{1+a}.$$

Integration by parts against  $d(\frac{1}{1-z})$  gives  $\Xi(w) = \Xi(1) + \frac{2c'}{1-c'}\chi_3(c') - 2Y(c')$  with  $Y(c') = \int_0^{c'} \frac{\chi_2(z)}{1-z} dz$ , and one more integration by parts (against  $-\ln(1-z)$ , using  $\chi_2'(z) = \frac{1}{2z} \ln \frac{1+z}{1-z}$ ) gives

$$Y(c') = -\ln(1-c')\chi_2(c') + \frac{1}{2} \int_0^{c'} \frac{\ln(1-z)\ln(1+z)}{z} dz - \frac{1}{2}\mathcal{C}(c'),$$

where both integrals are supplied by (A1) and (A4) (now with  $t(c') = \frac{1-c'}{1+c'} = a$ ). Assembling exactly as in §7 (the same log-collapse applies with  $a = \frac{1-c'}{1+c'}$ ) produces the companion formula displayed in **Result**; it was verified numerically to 60 digits at  $a = 0.1, 0.3, 0.5, 0.7, 0.9$ , and both branches give  $\frac{3\pi^2}{4}\ln 2 - \frac{21}{8}\zeta(3)$  at  $a = 1$ .

## 9. Specialization to $a = 2$ and $a = 3$

For  $a = 2$ :  $c = \frac{1}{3}$ ,  $\frac{1}{a} = \frac{1}{2}$ ,  $1 + c = \frac{4}{3}$ ,  $1 - c^2 = \frac{8}{9}$ ,  $\frac{a+1}{2a} = \frac{3}{4}$ ; for  $a = 3$ :  $c = \frac{1}{2}$ ,  $\frac{1}{a} = \frac{1}{3}$ ,  $1 + c = \frac{3}{2}$ ,  $1 - c^2 = \frac{3}{4}$ ,  $\frac{a+1}{2a} = \frac{2}{3}$ . The general formula then contains  $\text{Li}_{2,3}$  at  $\{\pm\frac{1}{3}, \pm\frac{1}{2}, \frac{3}{4}, \frac{8}{9}\}$  (for  $a = 2$ ) and  $\{\pm\frac{1}{2}, \pm\frac{1}{3}, \frac{3}{4}, \frac{2}{3}\}$  (for  $a = 3$ ). These are reduced with the following classical identities (each also verified numerically to  $\geq 40$  digits):

*Dilogarithm.* Reflection  $\text{Li}_2(x) + \text{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x)$ ; Landen  $\text{Li}_2(\frac{-x}{1-x}) = -\text{Li}_2(x) - \frac{1}{2} \ln^2(1-x)$ ; duplication  $\text{Li}_2(x) + \text{Li}_2(-x) = \frac{1}{2} \text{Li}_2(x^2)$ ;  $\text{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$ . From these one derives (used below):

$$\text{Li}_2\left(-\frac{1}{2}\right) = -\text{Li}_2\left(\frac{1}{3}\right) - \frac{1}{2} \ln^2 \frac{3}{2}, \quad \text{Li}_2\left(-\frac{1}{3}\right) = 2 \text{Li}_2\left(\frac{1}{3}\right) + \frac{1}{2} \ln^2 3 - \frac{\pi^2}{6},$$

$$\text{Li}_2\left(\frac{3}{4}\right) = 2 \text{Li}_2\left(\frac{1}{3}\right) - 2 \ln^2 2 + \ln^2 3, \quad \text{Li}_2\left(\frac{8}{9}\right) = \frac{\pi^2}{2} + 6 \ln 2 \ln 3 - 5 \ln^2 3 - 6 \text{Li}_2\left(\frac{1}{3}\right),$$

$$\text{Li}_2\left(\frac{2}{3}\right) = \frac{\pi^2}{6} - \ln \frac{1}{3} \ln \frac{2}{3} - \text{Li}_2\left(\frac{1}{3}\right) \quad (\text{reflection at } x = \frac{1}{3}).$$

For instance, the second identity follows from Landen at  $x = -\frac{1}{3}$  [ $\text{Li}_2(\frac{1}{4}) = -\text{Li}_2(-\frac{1}{3}) - \frac{1}{2} \ln^2 \frac{4}{3}$ ] combined with duplication and the  $\frac{1}{2}$ -values; the third from reflection at  $x = \frac{3}{4}$  plus duplication  $\text{Li}_2(\frac{1}{4}) = 2 \text{Li}_2(\frac{1}{2}) + 2 \text{Li}_2(-\frac{1}{2})$ ; the fourth from reflection at  $x = \frac{8}{9}$  plus duplication  $\text{Li}_2(\frac{1}{9}) = 2 \text{Li}_2(\frac{1}{3}) + 2 \text{Li}_2(-\frac{1}{3})$ .

*Trilogarithm.* Duplication  $\text{Li}_3(x) + \text{Li}_3(-x) = \frac{1}{4} \text{Li}_3(x^2)$ ; Landen's three-term relation (for  $0 < x < 1$ )

$$\text{Li}_3(x) + \text{Li}_3(1-x) + \text{Li}_3\left(\frac{-x}{1-x}\right) = \zeta(3) + \frac{\ln^3(1-x)}{6} + \frac{\pi^2}{6} \ln(1-x) - \frac{1}{2} \ln x \ln^2(1-x),$$

used at  $x = \frac{1}{4}$  (relating  $\text{Li}_3(\frac{3}{4}), \text{Li}_3(\frac{1}{4}), \text{Li}_3(-\frac{1}{3})$ ), at  $x = \frac{1}{9}$  (relating  $\text{Li}_3(\frac{8}{9}), \text{Li}_3(\frac{1}{9}), \text{Li}_3(-\frac{1}{8})$ ), and at  $x = \frac{1}{3}$  (relating  $\text{Li}_3(\frac{2}{3}), \text{Li}_3(\frac{1}{3}), \text{Li}_3(-\frac{1}{2})$ ); together with  $\text{Li}_3(\frac{1}{4}) = 4 \text{Li}_3(\frac{1}{2}) + 4 \text{Li}_3(-\frac{1}{2})$ ,  $\text{Li}_3(\frac{1}{9}) = 4 \text{Li}_3(\frac{1}{3}) + 4 \text{Li}_3(-\frac{1}{3})$  (duplication) and  $\text{Li}_3(\frac{1}{2}) = \frac{7\zeta(3)}{8} - \frac{\pi^2 \ln 2}{12} + \frac{\ln^3 2}{6}$ . (Landen's relation is itself proved by differentiating in  $x$  — the derivative identity reduces to the dilogarithm reflection and Landen formulas — and checking the value  $\zeta(3)$  at  $x \rightarrow 0^+$ .)

Carrying out these substitutions (exact rational bookkeeping, done symbolically and re-verified numerically to 100 digits) gives the two boxed evaluations in **Result**. Equivalently, with the Legendre chi, the  $a = 2$  value can be written

$$\begin{aligned} I(2, 1) &= \frac{21}{4} \zeta(3) + \frac{17}{2} \text{Li}_3\left(-\frac{1}{2}\right) - \frac{3}{2} \chi_3\left(\frac{1}{3}\right) - \frac{1}{4} \text{Li}_3\left(-\frac{1}{8}\right) - 2 \ln 2 \text{Li}_2\left(\frac{1}{3}\right) \\ &\quad + \pi^2 \left( \frac{3 \ln 3}{8} - \frac{7 \ln 2}{12} \right) + 2 \ln^2 2 \ln 3 - \ln 2 \ln^2 3 - \frac{7}{24} \ln^3 2. \end{aligned}$$

## Numerical verification

All computations with `mpmath` at 60–100 significant digits (`mp.dps = 60, 80, 100`), integrals by Gauss–Legendre `mp.quad` with split points  $[0, \frac{1}{50}, \frac{1}{5}, 1, 5, 50, 500, 5000, 10^5, \infty]$ .

1. **A identity (2.1):** agreement with direct quadrature to 60 digits at  $z = 0.3$  and  $z = 2.7$ .
2. **Key Lemma:**  $N(\gamma)$  vs. direct quadrature — 60 digits at  $\gamma = 0.5, 1, 2.3$ .
3. **Formula (1.1):** direct  $I(a, 1)$  vs.  $\int_0^{1/a} N_{\text{closed}}$  — 60 digits at  $a = 2, 3, 1.5, 5$ .

4. **General closed form** ( $a \geq 1$ ): vs. direct triple-arccot quadrature — **79–80 digits** of agreement at  $a = 1, 1.5, 2, 3, 5, 10$  (at working precision 80).
5. **Companion** ( $a < 1$ ): 59–60 digits at  $a = 0.1, 0.3, 0.5, 0.7, 0.9$ .
6. **Antiderivative lemmas (A1)–(A3), (2.1), and the  $H$ -derivative identity**: verified symbolically (sympy diff + simplify return 0).
7. **All dilog/trilog reduction identities of §9**: verified numerically to  $\geq 40$  digits.
8. **Final values (100 digits of working precision)**:

$$I(2, 1)_{\text{direct}} = I(2, 1)_{\text{closed}} = 1.51208605444427919260956727012207121505106065 \\ 1187506931662593594041998044060620041043 \dots$$

$$I(3, 1)_{\text{direct}} = I(3, 1)_{\text{closed}} = 1.25096386740952300916042368441067490079970979 \\ 1161621413290742488754988861633184783699 \dots$$

with  $|\text{direct} - \text{closed}| < 6 \cdot 10^{-101}$  in both cases (about 100 matching significant digits; conservatively  $\geq 79$  digits certified by the independent 80-dps run).

Structural exact checks: the general formula reproduces  $I(1, 1) = \frac{3\pi^2}{4} \ln 2 - \frac{21}{8}\zeta(3)$  exactly at  $a = 1$ , and the Key Lemma at  $\gamma = 1$  reproduces  $\int_0^{\pi/2} \theta^2 \cot \theta \, d\theta = \frac{\pi^2}{4} \ln 2 - \frac{7}{8}\zeta(3)$ .

## Notes

- **Scope.** The main boxed formula is proved for  $a \geq 1$ , which covers everything the question asks ( $I(a, 1)$  with  $a = 2, 3$ , and the general shape); the  $0 < a < 1$  companion is derived by the identical method (§8) and numerically confirmed, so  $I(a, 1)$  is in closed form for all  $a > 0$ . The answer to the posed question is therefore **yes**.
- **Rigor.** Every non-elementary evaluation rests on: differentiation under the integral sign with explicit dominating functions; Tonelli for one positive double integral; substitutions and integrations by parts with explicit antiderivatives, each verifiable (and machine-verified) by differentiation; and the classical polylogarithm identities listed in §9 (reflection/Landen/duplication/inversion-free set, plus the standard  $\text{Li}_{2,3}(\frac{1}{2})$  values). I am not aware of any gap.
- **Minimality of the basis is not claimed.** The constants  $\text{Li}_3(-\frac{1}{2})$ ,  $\text{Li}_3(\pm\frac{1}{3})$ ,  $\text{Li}_3(-\frac{1}{8})$ ,  $\text{Li}_2(\frac{1}{3})$  admit (to my knowledge) no further reduction to more elementary constants; other equivalent presentations (e.g. via  $\text{Li}_3(\frac{8}{9})$ ,  $\text{Li}_3(\frac{1}{9})$ ,  $\text{Li}_3(\frac{3}{4})$ , or Legendre chi values) are possible via the same identity set.
- The Key Lemma  $\int_0^\infty \frac{x \operatorname{arccot}^2 x}{x^2 + \gamma^2} dx = \frac{\pi^2}{4} \ln \frac{1+\gamma}{\gamma} - \frac{7}{8}\zeta(3) + \chi_3(\frac{\gamma-1}{\gamma+1})$  and the intermediate representation  $I(a, 1) = \int_0^{1/a} N$  make the structure of the whole family transparent; the same machinery would give the full two-parameter  $I(a, b)$  (an extra integration of the same kind), though this was not required here.