Cleo Bench Problem 17

Closer form for $\int_0^\infty \frac{(\arctan x)^2 \log^2(1+x^2)}{x^2} dx$

Derivation by Claude (Fable 5), closed-book*

July 2026

Problem

Evaluate in closed form

$$I \;=\; \int_0^\infty \frac{(\arctan x)^2\,\log^2(1+x^2)}{x^2}\,dx.$$

Result

$$\boxed{I \;=\; \frac{2}{3}\pi^3\ln 2 \;+\; \frac{4}{3}\pi\ln^3 2 \;+\; \frac{\pi\,\zeta(3)}{2} \;=\; \frac{\pi}{6}\Big(4\pi^2\ln 2 + 8\ln^3 2 + 3\zeta(3)\Big)}$$

Numerically

$$I = 17.6110991420260980345176855133221899447144462428948708760045\ldots$$

Derivation

Throughout, log denotes the principal branch on $\mathbb{C}\setminus(-\infty,0]$, and ln the real logarithm.

Step 0. Convergence

Write $f(x) = \dfrac{(\arctan x)^2\log^2(1+x^2)}{x^2}$. As $x\to 0^+$, $\arctan x \sim x$ and $\log(1+x^2)\sim x^2$, so $f(x) = O(x^4)$; as $x\to\infty$, $f(x) = O(\ln^2 x/x^2)$. Hence I converges absolutely.

Step 1. An algebraic splitting via $\log^4(1+ix)$

For $x>0$ the point $1+ix$ lies in the open first quadrant, so the principal logarithm gives

$$\log(1+ix) \;=\; \tfrac{1}{2}\ln(1+x^2) \;+\; i\arctan x \;=\; L+iT, \qquad L := \tfrac{1}{2}\ln(1+x^2), \quad T := \arctan x.$$

By the binomial theorem,

$$\Re\log^4(1+ix) = \Re\,(L+iT)^4 = L^4 - 6L^2T^2 + T^4,$$

hence

$$(\arctan x)^2\log^2(1+x^2) \;=\; T^2(2L)^2 \;=\; 4L^2T^2 \;=\; \frac{2}{3}\Big[L^4 + T^4 - \Re\log^4(1+ix)\Big].$$

Each of the three resulting integrals converges absolutely:

*Problem originally posed on Mathematics Stack Exchange (question 1153708, CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

- $\dfrac{L^4}{x^2} = O(x^6)$ at 0 and $O(\ln^4 x/x^2)$ at ∞;
- $\dfrac{T^4}{x^2} = O(x^2)$ at 0 and $O(x^{-2})$ at ∞;
- $\dfrac{|\log^4(1+ix)|}{x^2} = O(x^2)$ at 0 (since $\log(1+ix) = ix + O(x^2)$) and $O(\ln^4 x/x^2)$ at ∞ (since $|\log(1+ix)| \le \frac{1}{2}\ln(1+x^2) + \frac{\pi}{2}$).

Therefore, by linearity,

$$I = \frac{2}{3}\left[\frac{1}{16}P_1 + P_2 - \Re P_3\right], \qquad \begin{aligned} P_1 &:= \int_0^\infty \frac{\ln^4(1+x^2)}{x^2}\,dx, \\ P_2 &:= \int_0^\infty \frac{\arctan^4 x}{x^2}\,dx, \\ P_3 &:= \int_0^\infty \frac{\log^4(1+ix)}{x^2}\,dx, \end{aligned}$$

where $16L^4 = \ln^4(1+x^2)$ explains the factor $\frac{1}{16}$.

Step 2. $\Re P_3 = 0$ by contour rotation

Let $F(z) = \dfrac{\log^4(1+iz)}{z^2}$. The principal $\log(1+iz)$ fails to be analytic exactly where $1+iz \in (-\infty, 0]$, i.e. on the ray $\{it : \ t \ge 1\}$. Consequently F is analytic on the cut plane

$$W := \mathbb{C} \setminus \{iy : \ y \ge 0\},$$

which is open and **simply connected**, and which contains the closed fourth-quadrant sector $\{z \ne 0 : \ -\frac{\pi}{2} \le \arg z \le 0\}$ except that its two straight edges lie in W as well (the positive real axis and the negative imaginary axis avoid the cut).

For $0 < \epsilon < \frac{1}{2} < 2 < R$ consider the closed curve $\gamma \subset W$:

$$\gamma : \quad [\epsilon, R] \ \oplus \ A_R \ \oplus \ [-iR, -i\epsilon] \ \oplus \ a_\epsilon,$$

where $A_R : \phi \mapsto Re^{i\phi}$ with ϕ decreasing from 0 to $-\frac{\pi}{2}$, and $a_\epsilon : \phi \mapsto \epsilon e^{i\phi}$ with ϕ increasing from $-\frac{\pi}{2}$ to 0. By Cauchy's theorem in the simply connected domain W,

$$\int_\gamma F(z)\,dz = 0.$$

Arc estimates. For $z = Re^{i\phi}$, $\phi \in [-\frac{\pi}{2}, 0]$, the point $iz = Re^{i(\phi+\pi/2)}$ lies in the closed first quadrant, so $\Re(1+iz) \ge 1$, $0 \le \arg(1+iz) \le \frac{\pi}{2}$, and $|1+iz| \le 1+R$; hence

$$|\log(1+iz)| \le \ln(1+R) + \tfrac{\pi}{2}, \qquad \Big|\int_{A_R} F\Big| \le \frac{\pi R}{2}\cdot\frac{(\ln(1+R)+\pi/2)^4}{R^2} \xrightarrow[R\to\infty]{} 0.$$

For $|w| \le \frac{1}{2}$ one has $|\log(1+w)| = |\int_0^1 \frac{w\,dt}{1+tw}| \le \frac{|w|}{1-|w|} \le 2|w|$; hence on a_ϵ (where $|iz| = \epsilon \le \frac{1}{2}$), $|F| \le \frac{(2\epsilon)^4}{\epsilon^2} = 16\epsilon^2$ and

$$\Big|\int_{a_\epsilon} F\Big| \le \frac{\pi\epsilon}{2}\cdot 16\epsilon^2 \xrightarrow[\epsilon\to 0]{} 0.$$

The rotated ray. On $[-iR, -i\epsilon]$ put $z = -it$, $t \in [\epsilon, R]$: then $1+iz = 1+t > 0$, $z^2 = -t^2$, $dz = -i\,dt$, so

$$\int_{-iR}^{-i\epsilon} F(z)\,dz = \int_R^\epsilon \frac{\ln^4(1+t)}{-t^2}(-i)\,dt = -i\int_\epsilon^R \frac{\ln^4(1+t)}{t^2}\,dt.$$

Letting $\epsilon \to 0$, $R \to \infty$ (both limiting integrals converge: the integrand is $O(t^2)$ at 0 and $O(\ln^4 t/t^2)$ at ∞) yields

$$P_3 = \ i \int_0^\infty \frac{\ln^4(1+t)}{t^2}\, dt,$$

which is purely imaginary. Hence

$$\Re P_3 = 0.$$

(Cross-check, used only numerically below: substituting $s = 1/(1+t)$ and expanding $(1-s)^{-2} = \sum_{k\ge 1} k s^{k-1}$ — termwise by monotone convergence — gives $\int_0^\infty \frac{\ln^4(1+t)}{t^2}\, dt = \sum_{k\ge1} k \cdot \frac{4!}{k^5} = 24\,\zeta(4) = \frac{4\pi^4}{15}$, so $\Im P_3 = \frac{4\pi^4}{15}$; this was confirmed numerically, validating the rotation independently.)

Step 3. P_1 via a Beta-function derivative

Integrate by parts with $v = -1/x$; the boundary terms vanish because $\ln^4(1+x^2)/x = O(x^7)$ as $x \to 0$ and $O(\ln^4 x/x)$ as $x \to \infty$:

$$P_1 = \int_0^\infty \frac{1}{x} \cdot \frac{8x\,\ln^3(1+x^2)}{1+x^2}\, dx = 8 \int_0^\infty \frac{\ln^3(1+x^2)}{1+x^2}\, dx.$$

Substituting $x = \tan\theta$ (so $\frac{dx}{1+x^2} = d\theta$ and $\ln(1+\tan^2\theta) = -2\ln\cos\theta$),

$$P_1 = 8 \int_0^{\pi/2} (-2\ln\cos\theta)^3\, d\theta = -64 \int_0^{\pi/2} \ln^3 \cos\theta\, d\theta.$$

Let $f(a) = \int_0^{\pi/2} \cos^a\theta\, d\theta$ for $a > -1$. The substitution $u = \sin^2\theta$ gives the Wallis/Beta evaluation

$$f(a) = \tfrac{1}{2} B\Big(\tfrac12, \tfrac{a+1}{2}\Big) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\Big(\frac{a+1}{2}\Big)}{\Gamma\big(\frac{a}{2}+1\big)}.$$

For $a \in (-\frac12, \frac12)$ and $k \le 3$, $\left|\partial_a^k \cos^a\theta\right| = \cos^a\theta\,|\ln\cos\theta|^k \le \cos^{-1/2}\theta\,|\ln\cos\theta|^3 + 1$, which is integrable on $(0, \frac{\pi}{2})$; hence differentiation under the integral sign is justified and

$$f^{(3)}(0) = \int_0^{\pi/2} \ln^3\cos\theta\, d\theta.$$

Write $g = \ln f = \ln \frac{\sqrt{\pi}}{2} + \ln\Gamma\Big(\frac{a+1}{2}\Big) - \ln\Gamma\big(\frac{a}{2}+1\big)$. Then

$$g'(0) = \tfrac12[\psi(\tfrac12) - \psi(1)], \qquad g''(0) = \tfrac14[\psi'(\tfrac12) - \psi'(1)], \qquad g'''(0) = \tfrac18[\psi''(\tfrac12) - \psi''(1)].$$

From $\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k\ge0}(k+x)^{-n-1}$ and $\sum_{k\ge0}(k+\frac12)^{-s} = (2^s-1)\zeta(s)$:

$$\psi(\tfrac12) - \psi(1) = -2\ln 2, \qquad \psi'(\tfrac12) = \tfrac{\pi^2}{2},\ \psi'(1) = \tfrac{\pi^2}{6}, \qquad \psi''(\tfrac12) = -14\zeta(3),\ \psi''(1) = -2\zeta(3).$$

Hence

$$g'(0) = -\ln 2, \qquad g''(0) = \frac{\pi^2}{12}, \qquad g'''(0) = -\frac{3\zeta(3)}{2}.$$

Since $f = e^g$, $f''' = (g''' + 3g'g'' + g'^3)f$, and $f(0) = \frac{\pi}{2}$:

$$\int_0^{\pi/2} \ln^3\cos\theta\, d\theta = \frac{\pi}{2}\left[-\frac{3\zeta(3)}{2} - \frac{\pi^2\ln 2}{4} - \ln^3 2\right] = -\frac{3\pi\zeta(3)}{4} - \frac{\pi^3\ln 2}{8} - \frac{\pi\ln^3 2}{2}.$$

Therefore

$$P_1 = -64 \int_0^{\pi/2} \ln^3\cos\theta\, d\theta = 48\pi\zeta(3) + 8\pi^3\ln 2 + 32\pi\ln^3 2.$$

Step 4. P_2 **via Fourier analysis**

Substituting $x = \tan\theta$ (so $\frac{dx}{x^2} = \frac{\sec^2\theta}{\tan^2\theta}\,d\theta = \frac{d\theta}{\sin^2\theta}$),

$$P_2 = \int_0^{\pi/2} \frac{\theta^4}{\sin^2\theta}\,d\theta.$$

Integrating by parts with $\frac{1}{\sin^2\theta} = -\frac{d}{d\theta}\cot\theta$ (boundary terms: $\theta^4\cot\theta \sim \theta^3 \to 0$ at 0, and $\cot\frac{\pi}{2} = 0$):

$$P_2 = 4\int_0^{\pi/2} \theta^3\cot\theta\,d\theta.$$

Integrating by parts again with $\cot\theta = \frac{d}{d\theta}\ln\sin\theta$ (boundary terms: $\theta^3\ln\sin\theta \sim \theta^3\ln\theta \to 0$ at 0, and $\ln\sin\frac{\pi}{2} = 0$):

$$\int_0^{\pi/2} \theta^3\cot\theta\,d\theta = -3\int_0^{\pi/2} \theta^2\ln\sin\theta\,d\theta.$$

The system $\{\cos 2k\theta\}_{k\ge 0}$ is a complete orthogonal system in $L^2(0,\frac{\pi}{2})$ (it is the standard cosine basis for an interval of length $\frac{\pi}{2}$), with $\int_0^{\pi/2}\cos^2 2k\theta\,d\theta = \frac{\pi}{4}$ for $k \ge 1$. The expansion coefficients of $F(\theta) = \ln\sin\theta \in L^2(0,\frac{\pi}{2})$ are computed directly:

- $k = 0$: the classical Euler integral $\int_0^{\pi/2}\ln\sin\theta\,d\theta = -\frac{\pi}{2}\ln 2$. (Proof: with J denoting this integral, $\theta \mapsto \frac{\pi}{2} - \theta$ gives $J = \int_0^{\pi/2}\ln\cos\theta\,d\theta$; then $2J = \int_0^{\pi/2}\ln\frac{\sin 2\theta}{2}\,d\theta = \frac{1}{2}\int_0^{\pi}\ln\sin u\,du - \frac{\pi}{2}\ln 2 = J - \frac{\pi}{2}\ln 2$, using symmetry of $\ln\sin$ about $\frac{\pi}{2}$.)
- $k \ge 1$: integrating by parts,

$$\begin{aligned}\int_0^{\pi/2}\ln\sin\theta\,\cos 2k\theta\,d\theta &= \Big[\ln\sin\theta\,\frac{\sin 2k\theta}{2k}\Big]_0^{\pi/2} - \frac{1}{2k}\int_0^{\pi/2}\cot\theta\,\sin 2k\theta\,d\theta \\ &= -\frac{1}{2k}\cdot\frac{\pi}{2} = -\frac{\pi}{4k},\end{aligned}$$

  because the boundary term vanishes and $\cot\theta\,\sin 2k\theta = 1 + 2\sum_{j=1}^{k-1}\cos 2j\theta + \cos 2k\theta$ (an identity easily proved by induction from $\sin 2(k+1)\theta - \sin 2k\theta = 2\cos(2k+1)\theta\sin\theta$), whose integral over $(0,\frac{\pi}{2})$ equals $\frac{\pi}{2}$ since $\int_0^{\pi/2}\cos 2j\theta\,d\theta = 0$ for $j \ge 1$.

Thus the orthogonal expansion of $\ln\sin\theta$ in $L^2(0,\frac{\pi}{2})$ is

$$\ln\sin\theta = -\ln 2 - \sum_{k\ge 1}\frac{\cos 2k\theta}{k} \qquad \text{(convergence in } L^2(0,\tfrac{\pi}{2})\text{)},$$

by Riesz–Fischer (the coefficients are square-summable, and a function equals its expansion in a complete orthogonal system). Pairing with $\theta^2 \in L^2(0,\frac{\pi}{2})$ — legitimate term by term because the inner product is continuous in L^2 — and using the elementary moments

$$\int_0^{\pi/2}\theta^2\,d\theta = \frac{\pi^3}{24}, \qquad \int_0^{\pi/2}\theta^2\cos 2k\theta\,d\theta = \frac{(-1)^k\pi}{4k^2} \quad (k \ge 1),$$

(the latter from two integrations by parts, using $\sin\pi k = 0$, $\cos\pi k = (-1)^k$), we get

$$\int_0^{\pi/2}\theta^2\ln\sin\theta\,d\theta = -\frac{\pi^3\ln 2}{24} - \sum_{k\ge 1}\frac{1}{k}\cdot\frac{(-1)^k\pi}{4k^2} = -\frac{\pi^3\ln 2}{24} + \frac{\pi}{4}\eta(3) = -\frac{\pi^3\ln 2}{24} + \frac{3\pi\zeta(3)}{16},$$

where $\eta(3) = (1 - 2^{-2})\zeta(3) = \frac{3}{4}\zeta(3)$. Hence

$$\int_0^{\pi/2}\theta^3\cot\theta\,d\theta = \frac{\pi^3\ln 2}{8} - \frac{9\pi\zeta(3)}{16}, \qquad P_2 = \frac{\pi^3\ln 2}{2} - \frac{9\pi\zeta(3)}{4}.$$

Step 5. Assembly

$$\frac{P_1}{16} = 3\pi\zeta(3) + \frac{\pi^3 \ln 2}{2} + 2\pi \ln^3 2,$$

$$\frac{P_1}{16} + P_2 - \Re P_3 = \Big(3 - \tfrac{9}{4}\Big)\pi\zeta(3) + \Big(\tfrac{1}{2} + \tfrac{1}{2}\Big)\pi^3 \ln 2 + 2\pi \ln^3 2 = \frac{3\pi\zeta(3)}{4} + \pi^3 \ln 2 + 2\pi \ln^3 2,$$

and finally

$$I = \frac{2}{3}\left[\frac{3\pi\zeta(3)}{4} + \pi^3 \ln 2 + 2\pi \ln^3 2\right] = \frac{\pi\zeta(3)}{2} + \frac{2\pi^3 \ln 2}{3} + \frac{4\pi \ln^3 2}{3}.$$

$$\int_0^\infty \frac{(\arctan x)^2 \log^2(1+x^2)}{x^2}\, dx = \frac{2}{3}\pi^3 \ln 2 + \frac{4}{3}\pi \ln^3 2 + \frac{\pi\zeta(3)}{2}.$$

(In the notation of the original post: $\frac{a}{b} = \frac{2}{3}$ for $\pi^3 \ln 2$, $\frac{c}{d} = \frac{4}{3}$ for $\pi \ln^3 2$, the coefficient of $\zeta(3)$ is $\frac{\pi}{2}$, and all remaining guessed terms have coefficient 0.)

Numerical verification

All computations with mpmath at `mp.mp.dps = 90` (also run at 60); the integral was split as $\int_0^1 + \int_1^\infty$ with $x \mapsto 1/x$ on the tail (the transformed integrand has only an integrable $\ln^2$-singularity at 0, handled by tanh–sinh quadrature with an extra split point).

- Direct quadrature:

$$I = 17.611099142026098034517685513322189944714446242894870876004530241132223267 7\ldots$$

- Closed form $\frac{2\pi^3 \ln 2}{3} + \frac{4\pi \ln^3 2}{3} + \frac{\pi\zeta(3)}{2}$: identical string; $|\text{diff}| \approx 7.9 \cdot 10^{-90}$, i.e. agreement to about **89–90 significant digits** (all working digits).

- Piece checks (each to ≥ 40 digits): $P_1 = 48\pi\zeta(3) + 8\pi^3 \ln 2 + 32\pi \ln^3 2 = 386.680506\ldots$ ✓; $P_2 = \frac{\pi^3 \ln 2}{2} - \frac{9\pi\zeta(3)}{4} = 2.2491170739\ldots$ ✓; $\Re P_3 \approx -1.4 \cdot 10^{-57}$ (claim: 0) ✓; $\Im P_3 = 25.9757576090\ldots = 24\zeta(4) = \frac{4\pi^4}{15}$ ✓(independent confirmation of the contour rotation).

- Intermediates: $\int_0^{\pi/2} \ln^3 \cos\theta\, d\theta$ and $\int_0^{\pi/2} \theta^3 \cot\theta\, d\theta$ verified against their closed forms to ≥ 55 digits. The pointwise algebraic identity of Step 1 was checked at a sample point to 60 digits.

Notes

- The derivation is complete and self-contained. The one genuinely structural step is Step 1–2: the identity $4L^2T^2 = \frac{2}{3}[L^4 + T^4 - \Re \log^4(1+ix)]$ reduces I to two purely real classical integrals, because the mixed term $\Re \int_0^\infty x^{-2} \log^4(1+ix)\, dx$ vanishes identically after rotating the contour onto the negative imaginary axis (where $\log(1+ix)$ becomes the real $\ln(1+t)$). All contour, branch, and limiting arguments are justified in Step 2 (simply connected cut plane $W = \mathbb{C} \setminus i[0,\infty)$, explicit arc estimates).

- Every interchange of limit and integral is justified: differentiation under the integral sign in Step 3 (dominated by $\cos^{-1/2}\theta|\ln\cos\theta|^3$), and termwise integration in Step 4 via genuine L^2-convergence of the orthogonal expansion of $\ln\sin\theta$ (no Abel-summation or pointwise-Fourier subtleties are needed, since all expansion coefficients were computed by elementary integration by parts).
- No caveats: the numerical agreement is limited only by working precision.