

Cleo Bench Problem 13

Closed form for $\int_0^\infty \arctan\left(\frac{2\pi}{x - \ln x + \ln(\frac{\pi}{2})}\right) \frac{dx}{x+1}$

Derivation by Claude (Fable 5), closed-book*

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Problem

Evaluate in closed form

$$I = \int_0^\infty \arctan\left(\frac{2\pi}{x - \ln x + \ln \frac{\pi}{2}}\right) \frac{dx}{x+1}, \quad I \approx 3.38058252844534697939535922162769\dots$$

(The arctangent is the principal branch. Since $x - \ln x \geq 1$ on $(0, \infty)$, the argument of arctan is positive everywhere, so the integrand takes values in $(0, \pi/2)$.)

Result

$$I = \pi \ln \frac{\pi^2 + (1 - \ln \frac{\pi}{2})^2}{1 + \frac{\pi^2}{4}} = \pi \ln \frac{4\pi^2 + 4 \ln^2 \frac{2e}{\pi}}{4 + \pi^2}$$

Numerically $I = 3.3805825284453469793953592216276992165696856825906055108192183\dots$, matching the posted value; my closed form agrees with direct high-precision quadrature to **130 significant digits**.

Derivation

Throughout put

$$\phi_a(x) = x - \ln x + a \quad (x > 0), \quad a_0 = \ln \frac{\pi}{2} \approx 0.4516.$$

Since $\min_{x>0}(x - \ln x) = 1$ (attained at $x = 1$), we have $\phi_a(x) \geq 1 + a > 0$ for every $a \geq a_0$.

Step 1. A parameter family and differentiation under the integral

Define, for $a \geq a_0$,

$$I(a) = \int_0^\infty \arctan \frac{2\pi}{\phi_a(x)} \frac{dx}{x+1}, \quad \text{so } I = I(a_0).$$

*Problem originally posed on Mathematics Stack Exchange ([question 1150822](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

Convergence and the limit $a \rightarrow \infty$. Using $0 < \arctan y \leq \min(\frac{\pi}{2}, y)$ for $y > 0$ and $\phi_a \geq \phi_{a_0}$,

$$0 < \arctan \frac{2\pi}{\phi_a(x)} \cdot \frac{1}{x+1} \leq \frac{1}{x+1} \min\left(\frac{\pi}{2}, \frac{2\pi}{\phi_{a_0}(x)}\right) =: g(x).$$

g is integrable on $(0, \infty)$: near 0 it behaves like $\frac{2\pi}{-\ln x}$ (bounded), and as $x \rightarrow \infty$ like $2\pi/x^2$. Hence each $I(a)$ is finite, and since $\arctan \frac{2\pi}{\phi_a(x)} \rightarrow 0$ pointwise as $a \rightarrow \infty$, dominated convergence gives

$$\lim_{a \rightarrow \infty} I(a) = 0. \quad (1.1)$$

Differentiation in a . For fixed x , $\frac{\partial}{\partial a} \arctan \frac{2\pi}{\phi_a(x)} = -\frac{2\pi}{\phi_a(x)^2 + 4\pi^2}$, and for all $a \geq a_0$

$$\left| \frac{\partial}{\partial a} \arctan \frac{2\pi}{\phi_a(x)} \right| \frac{1}{x+1} \leq \frac{2\pi}{(x+1)(\phi_{a_0}(x)^2 + 4\pi^2)},$$

which is integrable in x (bounded near 0, $\sim 2\pi/x^3$ at ∞) and independent of a . By the standard dominated-convergence criterion, I is C^1 on $[a_0, \infty)$ with

$$I'(a) = -2\pi T(a), \quad T(a) := \int_0^\infty \frac{dx}{(x+1)[(x - \ln x + a)^2 + 4\pi^2]} (> 0). \quad (1.2)$$

By the fundamental theorem of calculus and (1.1) (the improper a -integral exists by monotone convergence, $T > 0$):

$$I = I(a_0) = 2\pi \int_{a_0}^\infty T(a) da. \quad (1.3)$$

Step 2. Closed form for $T(a)$ by a keyhole contour

Fix $a \geq a_0$. Let \log_* be the branch of the logarithm on $\Omega := \mathbb{C} \setminus [0, \infty)$ with $\arg z \in (0, 2\pi)$, and set

$$\Psi(z) = z - \log_* z + a, \quad z \in \Omega.$$

Ψ is holomorphic on Ω and extends continuously to each side of the cut: for $x > 0$,

$$\Psi(x + i0) = \phi_a(x) \text{ (real)}, \quad \Psi(x - i0) = \phi_a(x) - 2\pi i. \quad (2.1)$$

Since $\Im \frac{1}{u - 2\pi i} = \frac{2\pi}{u^2 + 4\pi^2}$ for real u , (2.1) gives the key jump identity

$$\Im \left[\frac{1}{\Psi(x + i0)} - \frac{1}{\Psi(x - i0)} \right] = -\frac{2\pi}{\phi_a(x)^2 + 4\pi^2}. \quad (2.2)$$

Lemma 1 (zeros of Ψ). *For every $a > -1$ the function Ψ has exactly one zero $\zeta(a)$ in Ω ; it is simple and equals*

$$\zeta(a) = \frac{v}{\sin v} e^{iv}, \quad \text{where } v = v(a) \in (0, \pi)$$

$$\text{is the unique solution of } m(v) := \frac{v}{\sin v} e^{-v \cot v} = e^a.$$

Moreover $\Im \zeta(a) = v(a)$ and $\Re \zeta(a) = v \cot v$. In particular, since $m(\pi/2) = \frac{\pi/2}{1} e^0 = \frac{\pi}{2} = e^{a_0}$,

$$\zeta(a_0) = \frac{i\pi}{2}. \quad (2.3)$$

Proof. Suppose $\Psi(\zeta) = 0$, $\zeta \in \Omega$. Splitting into real and imaginary parts ($\log_* \zeta = \ln |\zeta| + i \arg_* \zeta$, $\arg_* \zeta \in (0, 2\pi)$):

$$\ln |\zeta| = \Re \zeta + a, \quad \arg_* \zeta = \Im \zeta. \quad (2.4)$$

If $\Im \zeta \leq 0$: for ζ on the negative real axis $\arg_* \zeta = \pi \neq 0 = \Im \zeta$; for $\Im \zeta < 0$ we have $\arg_* \zeta \in (\pi, 2\pi)$ while $\Im \zeta < 0$ — in fact $\Im \Psi(\zeta) = \Im \zeta - \arg_* \zeta < -\pi$ throughout the open lower half-plane. So necessarily $\Im \zeta > 0$, whence $\arg_* \zeta = \text{Arg } \zeta \in (0, \pi)$ and $v := \Im \zeta = \text{Arg } \zeta \in (0, \pi)$. Writing $\zeta = r e^{iv}$, the identity $\Im \zeta = r \sin v = v$ forces $r = \frac{v}{\sin v}$, so $\zeta = \frac{v}{\sin v} e^{iv}$ and $\Re \zeta = v \cot v$. The modulus equation in (2.4) becomes $\ln \frac{v}{\sin v} = v \cot v + a$, i.e. $m(v) = e^a$. Conversely, if $m(v) = e^a$ with $v \in (0, \pi)$ and $\zeta = \frac{v}{\sin v} e^{iv}$, then $\log_* \zeta = \ln \frac{v}{\sin v} + iv$ and

$$\Psi(\zeta) = v \cot v + iv - \ln \frac{v}{\sin v} - iv + a = a - \ln m(v) = 0.$$

Uniqueness: on $(0, \pi)$,

$$\frac{d}{dv} \ln m(v) = \frac{1}{v} - 2 \cot v + v \csc^2 v = v + \frac{(1 - v \cot v)^2}{v} > 0,$$

(using $v \csc^2 v = v + v \cot^2 v$ and completing the square), so m is strictly increasing, with $m(0^+) = e^{-1}$ and $m(\pi^-) = +\infty$; thus $m : (0, \pi) \rightarrow (1/e, \infty)$ is a bijection and $e^a > 1/e$ has a unique preimage. Simplicity: $\Psi'(\zeta) = 1 - \frac{1}{\zeta} \neq 0$ because $\zeta \notin \mathbb{R}$. \square

(Equivalently $\zeta e^{-\zeta} = m(v) = e^a$, i.e. $\zeta(a) = -W_{-1}(-e^a)$ in Lambert- W notation; (2.3) is the classical $W(-\pi/2) = i\pi/2$. The Lambert- W language is not needed below.)

Keyhole integration. Consider $G(z) = \frac{1}{(z+1)\Psi(z)}$, meromorphic on Ω with simple poles only at $z = -1$ and $z = \zeta(a)$ (note $\Psi(-1) = -1 - i\pi + a \neq 0$ and $-1 \neq \zeta(a)$). Take the standard keyhole contour $\Gamma_{\epsilon, R, \eta}$: the segment just above the cut at height $+\eta$ traversed rightward from $|z| = \epsilon$ to $|z| = R$, the circle $|z| = R$ counterclockwise, the segment at height $-\eta$ leftward, and the circle $|z| = \epsilon$ clockwise ($0 < \eta \ll \epsilon < \frac{1}{2} < 2 < R$). Ω is simply connected and Γ is the positively oriented boundary of the enclosed region, which contains both poles ($\epsilon < 1 < R$ for $z = -1$; and $|\zeta(a)| = \frac{v}{\sin v} \geq 1$, $|\zeta(a)| < R$ for R large). The residue theorem gives

$$\oint_{\Gamma} G(z) dz = 2\pi i \left[\text{Res}_{z=-1} G + \text{Res}_{z=\zeta} G \right],$$

$$\text{Res}_{z=-1} G = \frac{1}{\Psi(-1)} = \frac{1}{a - 1 - i\pi} \quad (\log_*(-1) = i\pi), \quad \text{Res}_{z=\zeta} G = \frac{1}{(\zeta+1)\Psi'(\zeta)} = \frac{\zeta}{\zeta^2 - 1},$$

using $\Psi'(\zeta) = \frac{\zeta-1}{\zeta}$.

Arcs vanish. On $|z| = R$: $|\log_* z| \leq \ln R + 2\pi$, so $|\Psi(z)| \geq R - \ln R - 2\pi - a \geq R/2$ for large R , $|z+1| \geq R-1$; the contribution is $O(1/R) \rightarrow 0$. On $|z| = \epsilon$: $|\Psi(z)| \geq \ln \frac{1}{\epsilon} - \epsilon - a \rightarrow \infty$ and $|z+1| \geq 1 - \epsilon$; the contribution is $O(\epsilon / \ln \frac{1}{\epsilon}) \rightarrow 0$.

Edges converge to boundary values. On the upper edge $z = x + i\eta$: $\Re \Psi(z) = x - \ln |z| + a \geq x - \ln(x + \eta) + a \geq 1 + a - \eta > 0$, and also $\Re \Psi(z) \geq x - \ln(x+1) + a$, which grows linearly; on the lower edge, $\Im \Psi(z) = \Im z - \arg_* z < -\pi$, and again $|\Psi| \geq |\Re \Psi|$ gives linear growth for large x . Also $|z+1| \geq x+1$ on both edges. Hence the edge integrands are dominated, uniformly in η , by the integrable function $[(x+1) \max\{c_0, x - \ln(x+1) + a\}]^{-1}$ with $c_0 = \min\{1 + a - \epsilon, \pi\} > 0$, and they converge pointwise to the boundary values (2.1) as $\eta \rightarrow 0^+$. Dominated convergence, then $\epsilon \rightarrow 0^+$, $R \rightarrow \infty$ (the limiting integrals converge absolutely: the integrands are bounded near $x = 0$ and $O(x^{-2})$ at infinity) yield

$$\int_0^\infty \left[\frac{1}{\Psi(x+i0)} - \frac{1}{\Psi(x-i0)} \right] \frac{dx}{x+1} = 2\pi i \left[\frac{1}{a-1-i\pi} + \frac{\zeta(a)}{\zeta(a)^2-1} \right].$$

Taking imaginary parts and using (2.2):

$$-2\pi T(a) = 2\pi \Re \left[\frac{1}{a-1-i\pi} + \frac{\zeta(a)}{\zeta(a)^2-1} \right],$$

that is,

$$T(a) = \frac{1-a}{(1-a)^2+\pi^2} - \Re \frac{\zeta(a)}{\zeta(a)^2-1}. \quad (2.5)$$

(Verified numerically to 30+ digits at $a = a_0, 1, 3, 10$; see below.)

Step 3. Integrating over a : an exact antiderivative

Since $\Psi'(\zeta(a)) \neq 0$, the implicit function theorem shows $\zeta(a)$ is C^1 on $[a_0, \infty)$, and differentiating $\zeta - \log \zeta + a = 0$ (principal log; recall $\Im \zeta \in (0, \pi)$) gives

$$\zeta'(a) = \frac{-1}{1-\frac{1}{\zeta}} = \frac{\zeta}{1-\zeta}.$$

Consequently, since $1 + \zeta(a) \neq 0$,

$$\frac{d}{da} \ln |1 + \zeta(a)| = \Re \frac{\zeta'(a)}{1 + \zeta(a)} = \Re \frac{\zeta}{(1-\zeta)(1+\zeta)} = -\Re \frac{\zeta}{\zeta^2-1},$$

while $\frac{d}{da} \ln |a-1-i\pi| = \frac{a-1}{(a-1)^2+\pi^2}$. Comparing with (2.5): the function

$$B(a) := \ln \frac{|1 + \zeta(a)|}{|a-1-i\pi|} = \frac{1}{2} \ln \frac{(1 + \Re \zeta(a))^2 + (\Im \zeta(a))^2}{(a-1)^2 + \pi^2}$$

is an exact antiderivative: $B'(a) = T(a)$ on $[a_0, \infty)$. Hence by (1.3)

$$I = 2\pi \left[\lim_{A \rightarrow \infty} B(A) - B(a_0) \right]. \quad (3.1)$$

The limit at infinity is 0. Let $v = v(A) \uparrow \pi$ as $A \rightarrow \infty$ and write $\delta = \pi - v$, $s = -v \cot v = v \cot \delta > 0$, $\ell = \ln \frac{v}{\sin v} > 0$ (both positive for $v \in (\pi/2, \pi)$), so that $A = \ln m(v) = \ell + s$, $\Re \zeta = -s$, $\Im \zeta = v \in (\pi/2, \pi)$. For $\delta \leq \pi/4$:

$$s = v \cot \delta \leq \frac{\pi}{\tan \delta} \leq \frac{\pi}{\delta} \Rightarrow \delta \geq \frac{\pi}{s} \geq \frac{\pi}{A}, \quad \ell \leq \ln \frac{\pi}{\sin \delta} \leq \ln \frac{\pi}{(2/\pi)\delta} \leq \ln \frac{\pi A}{2} = O(\ln A),$$

using $\sin \delta \geq \frac{2}{\pi} \delta$ on $(0, \pi/2]$. Therefore $1 + \Re \zeta = 1 - s = 1 - A + \ell$ and

$$B(A) = \frac{1}{2} \ln \frac{(A-1-\ell)^2 + v^2}{(A-1)^2 + \pi^2} = \frac{1}{2} \ln \left[\left(1 - \frac{\ell}{A-1}\right)^2 \cdot \frac{1 + \frac{v^2}{(A-1-\ell)^2}}{1 + \frac{\pi^2}{(A-1)^2}} \right] \xrightarrow{A \rightarrow \infty} 0,$$

because $\ell/A = O(\ln A/A) \rightarrow 0$ and $v \leq \pi$. (Numerically $B(10^9) \approx -2.07 \times 10^{-8} \approx -\ln(10^9)/10^9$, matching this asymptotic.)

The value at a_0 . By (2.3), $\zeta(a_0) = i\pi/2$, so

$$B(a_0) = \ln \frac{|1 + \frac{i\pi}{2}|}{|\ln \frac{\pi}{2} - 1 - i\pi|} = \frac{1}{2} \ln \frac{1 + \frac{\pi^2}{4}}{(1 - \ln \frac{\pi}{2})^2 + \pi^2}.$$

Plugging into (3.1):

$$I = -2\pi B(a_0) = \pi \ln \frac{(1 - \ln \frac{\pi}{2})^2 + \pi^2}{1 + \frac{\pi^2}{4}}.$$

Since $1 - \ln \frac{\pi}{2} = \ln \frac{2e}{\pi}$, this can also be written

$$I = \pi \ln \frac{\pi^2 + \ln^2 \frac{2e}{\pi}}{1 + \frac{\pi^2}{4}} = \pi \ln \frac{4\pi^2 + 4 \ln^2 \frac{2e}{\pi}}{4 + \pi^2}. \quad \blacksquare$$

Remark. The problem's constants are precisely tuned: the shift $\ln \frac{\pi}{2}$ makes the relevant zero of $z - \log_* z + a$ land exactly at $z = i\pi/2$ (the classical identity $(\frac{i\pi}{2})e^{i\pi/2} = -\frac{\pi}{2}$, i.e. $W(-\pi/2) = i\pi/2$), and the numerator 2π is exactly the branch jump of the logarithm across the cut.

Numerical verification

All computations with `mpmath` (`verify.py` in the scratch directory), at 100–130 decimal digits.

1. **Direct evaluation.** Substituting $x = e^t$ (integrand then decays exponentially in both directions) and, independently, direct tanh–sinh quadrature in x with split points, both give at 100 dps:

$$I = 3.3805825284453469793953592216276992165696856825906055108192183062787494273960883\dots$$

The two methods agree to all 100 digits, and reproduce the 61 digits quoted in the problem (residual 6.3×10^{-63} = their truncation).

2. **Closed form.** $\pi \ln \frac{\pi^2 + (1 - \ln \frac{\pi}{2})^2}{1 + \pi^2/4}$ evaluated at 130 dps agrees with the 130-dps quadrature to relative error 1.3×10^{-131} : **130 significant digits matched.**
3. **Intermediate identities checked numerically.**
 - $\zeta(a_0) = i\pi/2$ to full precision; $\Psi_a(\zeta(a)) = 0$ and $\arg \zeta = \Im \zeta$ for $a = a_0, 1, 3, 10$.
 - Uniqueness of the zero: winding-number integral of Ψ'_a/Ψ_a around the rectangle $[-60, 60] \times [0.01, 2\pi - 0.01]$ equals 1.000... for $a = a_0$ and $a = 6$.
 - Residue formula (2.5) for $T(a)$: direct quadrature vs. formula agree to $\sim 5 \times 10^{-53}$ at $a = a_0, 1, 3, 10$.
 - Antiderivative: finite-difference derivative of $B(a)$ matches $T(a)$ to 25 digits at $a = 1, 5$.
 - Boundary term: $B(10^3), B(10^6), B(10^9) \approx -6.9 \times 10^{-3}, -1.38 \times 10^{-5}, -2.07 \times 10^{-8} \rightarrow 0$, consistent with $B(A) \sim -\ln A/A$.
 - Reassembly: $2\pi \int_{a_0}^{\infty} T(a) da$ (using (2.5)) reproduces I to 60 digits.

Notes

- The derivation is complete and self-contained: differentiation under the integral and the limits $a \rightarrow \infty$ are justified by explicit dominated-convergence bounds; the keyhole argument uses uniform lower bounds on $|\Psi|$ near both sides of the cut ($\Re \Psi \geq 1 + a - \epsilon$

above, $|\Im\Psi| > \pi$ below) plus linear growth at infinity, so the edge limits are legitimate; the zero-counting lemma is proved (not assumed) via the strict monotonicity of $m(v) = \frac{v}{\sin v} e^{-v \cot v}$; the boundary term at $a = \infty$ is controlled by explicit elementary inequalities.

- The only branch conventions used: \log_* with $\arg \in (0, 2\pi)$ on $\mathbb{C} \setminus [0, \infty)$ (so $\log_*(-1) = i\pi$, $\log_*(x - i0) = \ln x + 2\pi i$), and the principal arctan. No step relies on recalled literature values; the Lambert- W identification $\zeta(a) = -W_{-1}(-e^a)$ is mentioned only as a remark and cross-checked numerically.
- I am not aware of any gaps. Confidence: very high (rigorous derivation + 130-digit numerical agreement).