

# Cleo Bench Problem 4

Evaluate  $\int_0^{\pi/2} \frac{x^2 \log^2(\sin x)}{\sin^2 x} dx$

Derivation by Claude (Fable 5), closed-book\*

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## Problem

Evaluate in closed form

$$I = \int_0^{\pi/2} \frac{x^2 \log^2(\sin x)}{\sin^2 x} dx.$$

(The original post supplies the companion value  $\int_0^{\pi/2} \frac{x^2 \log(\sin x)}{\sin^2 x} dx = \pi \ln 2 - \frac{\pi}{2} \ln^2 2 - \frac{\pi^3}{12}$ ; we re-derive this from scratch along the way rather than assuming it.)

## Result

$$I = \frac{\pi \zeta(3)}{8} + \frac{\pi^3 \ln 2}{6} + \frac{\pi \ln^3 2}{3} + 2\pi \ln 2 - \pi \ln^2 2 - \frac{\pi^3}{6}$$

equivalently  $I = \frac{\pi \zeta(3)}{8} + \frac{\pi^3}{6}(\ln 2 - 1) + \frac{\pi \ln^3 2}{3} - \pi \ln^2 2 + 2\pi \ln 2$ . Numerically,

$$I = 2.08084652714326246266849090286004549406212100731588547858680778590266 \dots$$

## Derivation

Throughout,  $\log = \ln$  is the natural logarithm. All integrands below are continuous on  $(0, \pi/2]$ ; the only issue is the endpoint  $x = 0$ , where  $x/\sin x \rightarrow 1$ , so the integrand of  $I$  behaves like  $\log^2 x$ , which is absolutely integrable. Hence  $I$  converges (absolutely; the integrand is  $\geq 0$ ).

### Step 1. Integration by parts

On  $[\varepsilon, \pi/2]$  integrate by parts with

$$u = x^2 \log^2(\sin x), \quad dv = \frac{dx}{\sin^2 x}, \quad v = -\cot x,$$
$$du = (2x \log^2 \sin x + 2x^2 \cot x \log \sin x) dx.$$

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\*Problem originally posed on Mathematics Stack Exchange ([question 1142705](#), CC BY-SA), famously answered by user Cleo. This derivation was produced independently, offline, without access to the published answer, as part of the Cleo benchmark.

The boundary term is  $[-x^2 \log^2(\sin x) \cot x]_\varepsilon^{\pi/2}$ . At  $x = \pi/2$  it vanishes because  $\cot(\pi/2) = 0$  (and  $\log \sin(\pi/2) = 0$ ); as  $\varepsilon \rightarrow 0^+$ ,

$$\varepsilon^2 \log^2(\sin \varepsilon) \cot \varepsilon \sim \varepsilon \log^2 \varepsilon \rightarrow 0.$$

The two integrals produced converge absolutely (near 0:  $x \cot x \log^2 \sin x \sim \log^2 x$  and  $x^2 \cot^2 x \log \sin x \sim \log x$ ), so letting  $\varepsilon \rightarrow 0^+$ ,

$$I = 2 \underbrace{\int_0^{\pi/2} x \cot x \log^2(\sin x) dx}_{=:K} + 2 \int_0^{\pi/2} x^2 \cot^2 x \log(\sin x) dx.$$

Using  $\cot^2 x = \csc^2 x - 1$  and the (separate, absolute) convergence of both pieces,

$$\int_0^{\pi/2} x^2 \cot^2 x \log \sin x dx = \underbrace{\int_0^{\pi/2} \frac{x^2 \log \sin x}{\sin^2 x} dx}_{=:J} - \underbrace{\int_0^{\pi/2} x^2 \log \sin x dx}_{=:M},$$

so that

$$I = 2K + 2J - 2M. \quad (1)$$

We now compute  $K$ ,  $J$ ,  $M$  exactly.

## Step 2. A master formula: differentiating the Wallis integral

For  $\Re s > -1$  define

$$W(s) := \int_0^{\pi/2} \sin^s x dx.$$

Substituting  $t = \sin^2 x$  (so  $dx = \frac{1}{2} t^{-1/2} (1-t)^{-1/2} dt$ ) gives a Beta integral:

$$W(s) = \frac{1}{2} \int_0^1 t^{\frac{s-1}{2}} (1-t)^{-\frac{1}{2}} dt = \frac{1}{2} B\left(\frac{s+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(\frac{s+1}{2})}{2 \Gamma(\frac{s}{2} + 1)}. \quad (2)$$

Next, for  $s \in (-1, \infty)$  set

$$F(s) := \int_0^{\pi/2} x \cot x \sin^s x dx = \int_0^{\pi/2} x \cos x \sin^{s-1} x dx,$$

which converges absolutely (the integrand is  $\sim x^s$  at 0). Since  $\frac{d}{dx} \sin^s x = s \cos x \sin^{s-1} x$ , integration by parts on  $[\varepsilon, \pi/2]$  gives

$$s \int_\varepsilon^{\pi/2} x \cos x \sin^{s-1} x dx = \left[ x \sin^s x \right]_\varepsilon^{\pi/2} - \int_\varepsilon^{\pi/2} \sin^s x dx.$$

As  $\varepsilon \rightarrow 0^+$  the boundary term  $\varepsilon \sin^s \varepsilon \sim \varepsilon^{1+s} \rightarrow 0$  (as  $s > -1$ ), and all integrals converge, so

$$s F(s) = \frac{\pi}{2} - W(s), \quad s \in (-1, \infty) \quad (3)$$

(at  $s = 0$  both sides vanish, by  $W(0) = \pi/2$ ).

**Smoothness of  $F$  and differentiation under the integral.** Fix  $0 < \sigma < \frac{1}{2}$ . For  $|s| \leq \sigma$  and  $m \in \{0, 1, 2, 3\}$ ,

$$\left| x \cot x \sin^s x \log^m \sin x \right| \leq \sin^{-\sigma} x |\log \sin x|^m,$$

because  $0 < \sin x \leq 1$  makes  $s \mapsto \sin^s x$  bounded by  $\sin^{-\sigma} x$  for  $s \geq -\sigma$ , and  $x \cot x \leq 1$  on  $(0, \pi/2)$  (since  $\tan x \geq x$ ). By concavity  $\sin x \geq \frac{2}{\pi}x$  on  $[0, \pi/2]$ , so the dominating function is  $\leq Cx^{-\sigma}(|\log x| + c)^m$  near 0, which is integrable. Hence, by the standard theorem on differentiation under the integral sign (mean value theorem + dominated convergence, applied repeatedly),  $F \in C^3((-\sigma, \sigma))$  with

$$F^{(m)}(0) = \int_0^{\pi/2} x \cot x \log^m(\sin x) dx, \quad m = 0, 1, 2. \quad (4)$$

**Matching Taylor coefficients.**  $W$  is real-analytic in a neighbourhood of  $s = 0$  (by (2): both Gamma arguments are near  $\frac{1}{2}$  and 1, away from poles), and  $W(0) = \pi/2$ ; hence

$$G(s) := \frac{\pi/2 - W(s)}{s}$$

extends to a real-analytic function near 0. By (3),  $F = G$  on a punctured neighbourhood of 0, and both are continuous at 0, so  $F \equiv G$  near 0; consequently

$$F(0) = G(0), \quad F'(0) = G'(0), \quad F''(0) = G''(0), \quad (5)$$

i.e. the integrals (4) are read off from the Taylor expansion of  $W$  at  $s = 0$ .

### Step 3. Gamma-function input

We use the Weierstrass product  $\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n \geq 1} (1 + \frac{z}{n})e^{-z/n}$ , whose logarithmic derivative gives

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n \geq 0} \left( \frac{1}{n+1} - \frac{1}{n+z} \right), \quad \psi^{(m)}(z) = (-1)^{m+1} m! \sum_{n \geq 0} \frac{1}{(n+z)^{m+1}} \quad (m \geq 1),$$

the series for  $\psi^{(m)}$  obtained by termwise differentiation (uniform convergence on compacta avoiding the poles). Therefore:

- $\psi(1) = -\gamma$ .
- $\psi(1/2)$ : the partial sum  $\sum_{n=0}^{N-1} (\frac{1}{n+1} - \frac{2}{2n+1}) = H_N - (2H_{2N} - H_N) = 2H_N - 2H_{2N} \rightarrow -2 \ln 2$  (by  $H_N = \ln N + \gamma + o(1)$ ), so  $\psi(1/2) = -\gamma - 2 \ln 2$ .
- $\psi'(1) = \zeta(2) = \frac{\pi^2}{6}$ , and  $\psi'(1/2) = \sum_{n \geq 0} \frac{1}{(n+\frac{1}{2})^2} = 4 \sum_{n \geq 0} \frac{1}{(2n+1)^2} = 4(1 - \frac{1}{4})\zeta(2) = \frac{\pi^2}{2}$ .
- $\psi''(1) = -2\zeta(3)$ , and  $\psi''(1/2) = -2 \sum_{n \geq 0} \frac{1}{(n+\frac{1}{2})^3} = -16(1 - \frac{1}{8})\zeta(3) = -14\zeta(3)$ .

(Here  $\zeta(2) = \pi^2/6$  is Euler's classical evaluation; the odd-index sums use  $\sum_{\text{odd } m} m^{-p} = (1 - 2^{-p})\zeta(p)$ .)

Write  $W(s) = \frac{\pi}{2}e^{u(s)}$  with

$$u(s) = \ln \Gamma\left(\frac{s+1}{2}\right) - \ln \Gamma\left(\frac{s}{2} + 1\right) - \frac{1}{2} \ln \pi, \quad u(0) = 0.$$

Then

$$\begin{aligned} u'(0) &= \frac{1}{2}[\psi(\frac{1}{2}) - \psi(1)] = -\ln 2, & u''(0) &= \frac{1}{4}[\psi'(\frac{1}{2}) - \psi'(1)] = \frac{\pi^2}{12}, \\ u'''(0) &= \frac{1}{8}[\psi''(\frac{1}{2}) - \psi''(1)] = -\frac{3\zeta(3)}{2}. \end{aligned}$$

Exponentiating the cubic Taylor polynomial ( $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \dots$ ):

$$W(s) = \frac{\pi}{2} \left[ 1 - \ln 2 s + \left( \frac{\pi^2}{24} + \frac{\ln^2 2}{2} \right) s^2 - \left( \frac{\zeta(3)}{4} + \frac{\pi^2 \ln 2}{24} + \frac{\ln^3 2}{6} \right) s^3 + O(s^4) \right], \quad (6)$$

where the  $s^3$ -coefficient is  $\frac{1}{6}(u'''(0) + 3u'(0)u''(0) + u'(0)^3)$ .

#### Step 4. The three moments of $x \cot x$

From (6),

$$G(s) = \frac{\pi/2 - W(s)}{s} = \frac{\pi}{2} \left[ \ln 2 - \left( \frac{\pi^2}{24} + \frac{\ln^2 2}{2} \right) s + \left( \frac{\zeta(3)}{4} + \frac{\pi^2 \ln 2}{24} + \frac{\ln^3 2}{6} \right) s^2 + O(s^3) \right],$$

so by (4)–(5):

$$\int_0^{\pi/2} x \cot x \, dx = \frac{\pi \ln 2}{2}, \quad (7)$$

$$\int_0^{\pi/2} x \cot x \log \sin x \, dx = -\frac{\pi^3}{48} - \frac{\pi \ln^2 2}{4}, \quad (8)$$

$$K = \int_0^{\pi/2} x \cot x \log^2 \sin x \, dx = 2! \cdot \frac{\pi}{2} \left( \frac{\zeta(3)}{4} + \frac{\pi^2 \ln 2}{24} + \frac{\ln^3 2}{6} \right) = \frac{\pi \zeta(3)}{4} + \frac{\pi^3 \ln 2}{24} + \frac{\pi \ln^3 2}{6}. \quad (9)$$

#### Step 5. The elementary companions, and the hinted integral $J$

Integrating by parts exactly as in Step 1 (boundary terms vanish by the same estimates):

- With  $u = x^2$ :  $\int_0^{\pi/2} \frac{x^2}{\sin^2 x} \, dx = \left[ -x^2 \cot x \right]_0^{\pi/2} + 2 \int_0^{\pi/2} x \cot x \, dx \stackrel{(7)}{=} \pi \ln 2$ , hence

$$\int_0^{\pi/2} x^2 \cot^2 x \, dx = \pi \ln 2 - \int_0^{\pi/2} x^2 \, dx = \pi \ln 2 - \frac{\pi^3}{24}. \quad (10)$$

- With  $u = x^2 \log \sin x$  (boundary term  $\sim x \log x \rightarrow 0$  at 0):

$$J = \int_0^{\pi/2} \frac{x^2 \log \sin x}{\sin^2 x} \, dx = 2 \int_0^{\pi/2} x \cot x \log \sin x \, dx + \int_0^{\pi/2} x^2 \cot^2 x \, dx \stackrel{(8),(10)}{=} \pi \ln 2 - \frac{\pi \ln^2 2}{2} - \frac{\pi^3}{12}. \quad (11)$$

This reproduces the value quoted in the problem statement, confirming the machinery.

#### Step 6. $M = \int_0^{\pi/2} x^2 \log \sin x \, dx$ via the Fourier series of $\log(2 \sin x)$

For  $0 < x < \pi$ ,

$$1 - e^{2ix} = -2i \sin x e^{ix} = 2 \sin x e^{i(x-\pi/2)},$$

and since  $2 \sin x > 0$ ,  $x - \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , the principal logarithm satisfies

$$\operatorname{Log}(1 - e^{2ix}) = \log(2 \sin x) + i \left( x - \frac{\pi}{2} \right).$$

The series  $-\sum_{k \geq 1} z^k/k$  converges for  $|z| = 1$ ,  $z \neq 1$  (Dirichlet's test, as the partial sums of  $\sum z^k$  are bounded), and by Abel's theorem its value is  $\lim_{r \rightarrow 1^-} (-\operatorname{Log}(1 - rz)) = -\operatorname{Log}(1 - z)$ ; the limit uses continuity of  $\operatorname{Log}$  on the right half-plane, where  $1 - rz$  stays for  $0 \leq r \leq 1$  because  $\Re z < 1$ . Taking real parts at  $z = e^{2ix}$ :

$$\log(2 \sin x) = -\sum_{k=1}^{\infty} \frac{\cos 2kx}{k} \quad \text{pointwise on } (0, \pi). \quad (12)$$

**Termwise integration against  $x^2$ .** On  $(0, \pi)$  the system  $\{\cos 2kx\}_{k \geq 1}$  is orthogonal with  $\|\cos 2kx\|_{L^2}^2 = \pi/2$ ; since  $\sum k^{-2} < \infty$ , the partial sums  $S_N(x) = \sum_{k \leq N} \frac{\cos 2kx}{k}$  form a Cauchy

sequence in  $L^2(0, \pi)$  and converge to some  $f \in L^2$ . A subsequence converges a.e. to  $f$ ; by (12) the full sequence converges pointwise to  $-\log(2 \sin x)$ , so  $f = -\log(2 \sin x)$  a.e. With  $\phi = x^2 \chi_{(0, \pi/2)} \in L^2(0, \pi)$ , Cauchy–Schwarz gives

$$\int_0^{\pi/2} x^2 S_N(x) dx \longrightarrow - \int_0^{\pi/2} x^2 \log(2 \sin x) dx.$$

The cosine moments are elementary: differentiating  $\frac{x^2 \sin 2kx}{2k} + \frac{x \cos 2kx}{2k^2} - \frac{\sin 2kx}{4k^3}$  returns  $x^2 \cos 2kx$ , and evaluating between 0 and  $\pi/2$  (where  $\sin k\pi = 0$ ,  $\cos k\pi = (-1)^k$ ),

$$\int_0^{\pi/2} x^2 \cos 2kx dx = \frac{(-1)^k \pi}{4k^2}.$$

Hence

$$\int_0^{\pi/2} x^2 S_N dx = \frac{\pi}{4} \sum_{k \leq N} \frac{(-1)^k}{k^3} \longrightarrow -\frac{\pi}{4} \eta(3) = -\frac{\pi}{4} \cdot \frac{3}{4} \zeta(3) = -\frac{3\pi \zeta(3)}{16},$$

using  $\eta(3) = \sum_{k \geq 1} (-1)^{k-1} k^{-3} = (1 - 2^{-2}) \zeta(3)$  (split even/odd indices). Therefore

$$\int_0^{\pi/2} x^2 \log(2 \sin x) dx = \frac{3\pi \zeta(3)}{16}, \quad M = \frac{3\pi \zeta(3)}{16} - \ln 2 \cdot \frac{\pi^3}{24}. \quad (13)$$

## Step 7. Assembly

Insert (9), (11), (13) into (1):

$$\begin{aligned} I &= 2K + 2J - 2M \\ &= \left( \frac{\pi \zeta(3)}{2} + \frac{\pi^3 \ln 2}{12} + \frac{\pi \ln^3 2}{3} \right) + \left( 2\pi \ln 2 - \pi \ln^2 2 - \frac{\pi^3}{6} \right) - \left( \frac{3\pi \zeta(3)}{8} - \frac{\pi^3 \ln 2}{12} \right) \\ &= \frac{\pi \zeta(3)}{8} + \frac{\pi^3 \ln 2}{6} + \frac{\pi \ln^3 2}{3} + 2\pi \ln 2 - \pi \ln^2 2 - \frac{\pi^3}{6}. \\ \int_0^{\pi/2} \frac{x^2 \log^2(\sin x)}{\sin^2 x} dx &= \frac{\pi \zeta(3)}{8} + \frac{\pi^3 \ln 2}{6} + \frac{\pi \ln^3 2}{3} + 2\pi \ln 2 - \pi \ln^2 2 - \frac{\pi^3}{6}. \end{aligned}$$

## Numerical verification

All computations with `mpmath` (scripts `check_main.py`, `check_digits.py`, `check_gl.py` in the work directory `/tmp/claude-1000/-home-riv-Code-cleo-bench/817bd907-85a0-4bc2-a728-b3544bb304b6/scratchpad/cleo/work/q1142705-evaluate-int-0-pi-2-frac-x-2-log-2-sin-x-sin-2x/`).

- **Main integral, tanh–sinh quadrature, `mp.dps = 90`**, split points  $\{0, 10^{-3}, 10^{-1}, \pi/4, \pi/2\}$ :
  - direct integral:  
2.08084652714326246266849090286004549406212100731588547858680778590265996293...
  - closed form:  
2.08084652714326246266849090286004549406212100731588547858680778590265996293...
  - $|\text{difference}| = 4.9 \times 10^{-91}$  — agreement to **90 significant digits** (full working precision).
- **Independent method:** Gauss–Legendre quadrature at `mp.dps = 60` on  $[10^{-40}, \pi/2]$  with geometric split points  $10^{-40}, 10^{-39}, \dots, 10^{-1}, \pi/4, \pi/2$ , plus the analytic tail  $\int_0^\varepsilon \log^2 x dx = \varepsilon(\log^2 \varepsilon - 2 \log \varepsilon + 2)$  for  $\varepsilon = 10^{-40}$  (the neglected correction is  $O(\varepsilon^3 \log^2 \varepsilon) \approx 10^{-117}$ ): difference from the closed form = 0.0 at 60-digit precision.

- **Every intermediate identity was also checked numerically at 60 digits** (differences 0 or  $\lesssim 10^{-61}$ ):  $K$  from (9); (8); (7);  $J$  from (11);  $M$  from (13);  $\int_0^{\pi/2} x^2 \csc^2 x dx = \pi \ln 2$ ; and the master identity (3) at the off-center points  $s = 0.3$  (agreement  $\sim 10^{-61}$ ) and  $s = -0.4$  (agreement  $\sim 10^{-39}$ , limited only by quadrature near the  $x^s$  endpoint singularity).

## Notes

- The derivation is complete and self-contained up to the following classical facts used without proof: the Beta–Gamma identity  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ ; the Weierstrass product for  $\Gamma$  (equivalently, the partial-fraction series for  $\psi$  and its derivatives); Euler’s  $\zeta(2) = \pi^2/6$ ; Abel’s theorem on power series; and completeness of  $L^2$  together with the fact that  $L^2$ -convergence yields an a.e.-convergent subsequence. All interchanges of limit and integral are justified in the text (dominated convergence with explicit dominating functions in Step 2;  $L^2$ /Cauchy–Schwarz in Step 6), and every improper integral and boundary term is handled by explicit asymptotics.
- The identity (3) plus the expansion (6) is the engine of the solution: it delivers all moments  $\int_0^{\pi/2} x \cot x \log^m \sin x dx$  at once, and in particular re-derives the value of  $J$  quoted in the problem statement — an internal consistency check on the method, independently confirmed numerically.
- No caveats or gaps remain; the closed form is verified to 90 significant digits by two different quadrature schemes.